HOLES AND DOMINOES IN MEYNIEL GRAPHS

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ABSTRACT

Many practical problems (frequency assignment, register allocation, timetables) may be formulated as graph (vertex-)coloring problems, but finding solutions for them is difficult as long as they are treated in the most general case (where the graph is arbitrary), since vertex coloring has been proved to be NP-complete. The problem becomes much easier to solve if the graph resulting from the modelisation of the practical application belongs to some particular class of graphs, for which solutions to the problem are known. Meyniel graphs form such a class (a fast coloring algorithm can be found in [9]), for which an efficient recognizing algorithm is therefore needed.

A graph $G = (V, E)$ is said to be a Meyniel graph if every odd cycle of $G$ on at least five vertices contains at least two chords. Meyniel graphs generalize both $i$-triangulated and parity graphs, two well known classes of perfect graphs that will be present in our paper in Section 7.

In [2], Burlet and Fonlupt propose a characterization of Meyniel graphs which relies on the following property: the class of Meyniel graphs may be obtained from some basic Meyniel graphs using a binary operation called amalgam. Besides the theoretical interest of this result, a practical interest arises because of the polynomial recognition algorithm which can be obtained. Unfortunately, it is quite expensive to verify if a given graph is the amalgam of two graphs (therefore the complexity of the whole algorithm is in $O(n^7)$), and this supports the idea that a new point of view is needed to find a more efficient algorithm.

Our approach of Meyniel graphs will be directed through the search of a general structure. Intuitively, a Meyniel graph either will be simple (i.e. with no hole or domino), or will have a skeleton around which the rest of the graph will be regularly organized. As suggested, the first type of Meyniel graphs is simple to identify. For the second type, a deeper analysis is necessary; it yields a characterization theorem, which is used to deduce the $O(m^2 \cdot mn)$ recognition algorithm.
1. Introduction

Meyniel [11] proved that graphs whose odd cycles (of length five or more) have at least two chords are perfect, that is, for each of them induced subgraphs the clique number equals the chromatic number. Several years later, Grötschel, Lovász and Schrijver’s [8] showed that the NP-problems of computing the chromatic, clique, stability and clique covering number become polynomial on perfect graphs. Together, these two results asserted the necessity to investigate the algorithmic aspects of the graphs now called Meyniel graphs.

We are interested here in the recognition of these graphs, for which we give a characterization theorem and a recognition algorithm which improves a lot the complexity of the previous one.

The reader is supposed familiar with the classical notions of (elementary) path and cycle, parity of a path or a cycle, connected and 2-connected graph, connected and 2-connected component. For these notions, as well as for finding linear algorithms to compute the (2-)connected components of a graph, see [14].

The subpath $P_{x_i,x_j}$ ($i < j$) of a path $P = [x_1, \ldots, x_{k+1}]$ is the path $[x_i, x_{i+1}, \ldots, x_j]$. The similar notation will be used for a subpath of a cycle. Special attention will be paid to induced (or chordless) paths and cycles (i.e. which have no edges but the ones indicated before), denoted respectively by $P_k$ and $C_k$ if they have $k$ vertices.

Several particular graphs will often be used along the presentation: the hole (chordless cycle of length at least equal to 5), the domino (cycle on six vertices with a unique chord joining two vertices at distance three along the cycle) and the house (cycle on five vertices with a unique chord).

We close this section by giving some notations. All the vertices adjacent in $G$ to a fixed vertex $v$ will form the set $N_G(v)$ (or $N(v)$ if no confusion is possible) called neighbourhood of $v$. If either $A \subseteq V$ or $A$ is an induced subgraph of $G$, then $G - A$ will denote the subgraph of $G$ induced by all the vertices not in $A$. Also, if $H$ is an induced subgraph of $G$ and either $A \subseteq V$ or $A$ is an induced subgraph of $G$, then $H + A$ will be the subgraph induced in $G$ by the vertices in $H$ and the vertices in $A$. One more simplification will be made if $A = \{x\}$: we will write $G - x$, $H + x$ instead of $G - \{x\}$, $H + \{x\}$.

Modular decomposition

A set $M \subseteq V$ of vertices is said to be a module in $G = (V,E)$ if every vertex in $V \setminus M$ is either adjacent to all the vertices in $M$, or to none of them. A module is trivial if it is the empty set, a singleton or $V$ itself, and a graph with more than 2 vertices is prime if all its modules are trivial.

The modular decomposition associates with an arbitrary graph $G$ a unique decomposition tree $T(G)$, whose leaves are the vertices of $G$, while the internal nodes are modules labeled with $P$, $S$ and $N$ (for parallel, serial and neighbourhood module; see [7], [10] or [5] for details). The root of $T(G)$ corresponds to the entire $G$, which is uniquely decomposed in modules (represented in $T(G)$ by the children of the root); each module is also decomposed in submodules and so on, until only
singletons may be found. At the end, if \( r \) is a node of \( T(G) \) (not a leaf) with children \( r_1, r_2, \ldots, r_k \), the graph with vertices \( r_1, r_2, \ldots, r_k \) and edges \( r_i r_j \) such that any vertex in \( M(r_i) \) is adjacent to any vertex in \( M(r_j) \) is called the representative graph of \( r \) (then for a \( P \)-node, an \( S \)-node, an \( N \)-node the representative graph is respectively a stable set, a clique and a prime graph).

The unicity of the decomposition may be found in [7]. As shown by McConnel, Spinrad [10] and independently by Courner, Habib [5], the modular decomposition tree of any graph is computable in linear (in number of edges) time.

2. Preliminary aspects

Consider a Meyniel graph \( G \) (that we can suppose connected) and a hole \( C \) (obviously of even length) of \( G \). Try to identify all the positions an arbitrary vertex \( w \) out of \( C \) can occupy with respect to \( C \).

**Lemma 1** If \( G \) is a Meyniel graph and \( C \) is an induced cycle of \( G \), then for any \( w \in V - V(C) \) we have exactly one of the following statements:

1. \( w \) is adjacent to no vertex of \( C \);
2. \( w \) is adjacent to exactly one vertex of \( C \);
3. \( C + w \) is a bipartite 2-connected graph;
4. \( w \) is adjacent to every vertex of \( C \).
5. \( w \) is a copy of a certain \( z_i \) (with respect to \( C \)), i.e. \( w \) is exclusively adjacent to \( z_{i-1}, z_i, z_{i+1} \) (with the convention \( z_0 = z_k, z_{k+1} = z_1 \)).

**Proof.** It is easy to see that any pair of statements cannot be simultaneously true. Let \( C = [z_1, z_2, \ldots, z_k, z_1] \) \((k \geq 6)\) and suppose, by contradiction, that none of the statements 1, 2, 3 is verified. Then \( w \) forms at least one odd cycle with \( C \) (of length three or more) therefore we can apply the following result to deduce that \( w \) is adjacent to at least two consecutive vertices of \( C \).

**Claim 1** [11] Let \( G \) be a Meyniel graph, \( w \) a vertex of \( G \) and \( u, v \) two non-adjacent neighbours of \( w \), joined in \( G \) by an odd chordless path \( P \). Then \( V(P) \subseteq N(w) \).

We finish the proof of Lemma 1 using this corollary of Proposition 6 in [2]:

**Claim 2** Let \( C = [z_1, z_2, \ldots, z_p, z_1] \) be an induced cycle of length at least four of the Meyniel graph \( G \) and \( w \in V - V(C) \) a vertex adjacent to two consecutive vertices (say \( z_1, z_2 \)) on \( C \). Then \( w \) is a copy of \( z_1 \), or a copy of \( z_2 \), or is adjacent to every vertex of \( C \).

And now Lemma 1 is proved.

This lemma suggests a possibility to express the property of the graph to be “Meyniel-like” around a cycle, by putting the vertices in sets with particular properties. Unfortunately, this approach is not really the best one because of the instability of the partition in sets: even a very small change of the cycle (replacing a vertex with one of its copies) implies great changes in the sets; therefore it is very difficult to exploit the results obtained for one cycle when considering another one. That is why it is preferable to bring together all the copies of a given cycle \( C \) and then define all the other sets taking into account all these copies.
Two more problems appear here: firstly, the induced cycle on four vertices has a particular behaviour (a vertex \( w \) adjacent to all the vertices of a \( C_4 \) is not necessarily adjacent to the copies of the vertices, while for a \( C_k, k \geq 6 \), this is true); secondly, identifying an induced cycle on at least six vertices (i.e. a hole) is not easy (from an algorithmical point of view). A solution to these problems is to find either a hole or a domino in our graph, since for dominoes a statement equivalent to Lemma 1 holds (see below).

Now, we have a rough representation of the Meyniel graph, which we try to refine following the two steps below (details are found respectively in Sections 3, 4):

**Step I** augmenting the initial cycle or domino to a maximal 2-connected bipartite subgraph \( \Gamma \) (called skeleton of \( G \)) for which Lemma 1 is still true; putting together all the copies of \( \Gamma \) and defining the other sets with respect to all these copies.

**Step II** improving \( \Gamma \) using a refining operation.

Two kinds of Meyniel graphs are then found:
- the so-called HHD-graphs, containing no hole, no house and no domino;
- the Meyniel graphs which have an almost regular structure around a 2-connected bipartite graph on at least six vertices (the skeleton).

We now start to give the details of the announced steps.

### 3. Building a large skeleton

Let \( G = (V, E) \) be a Meyniel graph containing a hole or a domino (called \( C \)) and \( \Gamma \) a bipartite 2-connected subgraph of \( G \) containing \( C \). Any vertex \( z \) in \( \Gamma \) can have copies in \( G \) (with respect to \( \Gamma \)), that is, vertices \( w \) such that \( N_{\Gamma}(w) \setminus \{z\} = N_{\Gamma}(z) \). In the graph induced in \( G \) by the copies of \( z \) (\( z \) included), denote \( D_{\Gamma}(z) \) the connected component of \( z \) and notice that \( D_{\Gamma}(z) \cap V(\Gamma) = \{z\} \) (otherwise a vertex can be found in \( D_{\Gamma}(z) \) with a too large neighbourhood in \( \Gamma \)). So \( z \) may be seen as the representative vertex of all its copies, while the graph \( \Gamma_D \) induced by \( \cup_{z \in \Gamma} D_{\Gamma}(z) \) may be seen as the graph containing all the “near” copies of \( \Gamma \) (see Fig. 1).

The set of vertices adjacent to all the vertices in \( C \) (recall that \( \Gamma \) contains \( C \)) is denoted \( T \). In \( G - T \), denote \( S \) all the vertices which are not in the same connected component as \( C \).

Till now, we defined the sets of vertices which correspond to the statements 4, 5 and a part of the set corresponding to 1. As indicated, it is preferable to define the rest of the sets (corresponding to 1, 2, 3) taking into account all the “near” copies of \( \Gamma \), i.e. all the graph \( \Gamma_D \). At the beginning, they will be mixed up, but progressively they will be separated as needed. For the moment, define (see Fig. 1, where \( z \) is an arbitrary vertex of \( \Gamma \)):

- \( C_{\Gamma}(z) = \{w \in V - V(\Gamma_D) - T \mid w \text{ is adjacent to all the vertices in } D_{\Gamma}(z)\} \)
- \( W_{\Gamma}(z) = \{w \in V - V(\Gamma_D) - C_{\Gamma}(z) - T \mid w \text{ is adjacent to some vertex in } D_{\Gamma}(z)\} \)
- \( L_{\Gamma}(z) = \{w \in V - C_{\Gamma}(z) - W_{\Gamma}(z) - D_{\Gamma}(z) - T \mid w \text{ is connected to } C_{\Gamma}(z) \text{ or to } W_{\Gamma}(z) \text{ in } G - D_{\Gamma}(z) - T\} \)

Obviously, for any connected bipartite subgraph \( \Gamma \) of \( G \), this decomposition of \( G \) around \( \Gamma \) is unique, but the indicated sets do not partition the graph.
With these definitions and with the notation $T_H$ for the set of vertices adjacent to all the vertices in a subgraph $H$, we can now reformulate (and slightly generalize) Claim 2 in the following way:

**Claim 3** Let $H$ be either an induced cycle of length at least four or a domino of the Meyniel graph $G$ and $w \in V - V(H)$ a vertex which has two adjacent neighbours (say $z, z'$) on $H$. Then $w \in D_H(z) \cup D_H(z') \cup T_H$.

**Proof.** If $H$ is an induced cycle, the result is immediate from Claim 2. If it is a domino, we can apply Claim 2 for one of the two $C_4$ in $H$. If $w$ is adjacent to every vertex of this $C_4$, then again by Claim 2 we must have $w \in T_H$; otherwise it is easy to see that $w$ must be the copy (with respect to $H$) of a vertex of $H$. □

**Remark 1** Notice that Claim 3 is not true for any bipartite graph. To see this, it is sufficient to consider a cycle on six vertices with two chords joining vertices at distance 3 along the cycle. There can be a vertex adjacent only to the vertices of degree 3.

Now, a subgraph $\Gamma$ of $G$ is called a skeleton of $G$ (with respect to the hole or domino $C$) if the following conditions are verified:

1. $\Gamma$ is bipartite 2-connected and contains $C$;
2. for every $z \in V(\Gamma)$, $D_\Gamma(z)$ is a module in $\Gamma_D$;
3. for every $z \in V(\Gamma)$, $W_\Gamma(z)$ is not connected to $\Gamma_D$ in $G - D_\Gamma(z) - T$.

If these conditions hold, in $G - T - S$ the sets $W_\Gamma(z), C_\Gamma(z), L_\Gamma(z)$ may be seen as the irregular parts of a quite regular structure. In fact (see the following lemma), as long as there exist some sets $C_\Gamma(z)$ connected to $\Gamma_D$ in $G - D_\Gamma(z) - T$, the regular structure may be augmented (i.e. the skeleton may be grown). Finally, no $C_\Gamma(z)$ will be connected to $\Gamma_D$ in $G - D_\Gamma(z) - T$, i.e. $L_\Gamma(z)$ will contain no vertex of $\Gamma_D$, so the irregular parts will be represented by some sets $T_\Gamma(z) = W_\Gamma(z) \cup C_\Gamma(z) \cup L_\Gamma(z)$ “attached” to the corresponding $D_\Gamma(z)$ and adjacent to no other vertices but, possibly, the ones in $T$. In that case, the sets $D_\Gamma(z), I_\Gamma(z), T, S$ will partition $V$.

**Lemma 2** Let $G$ be a Meyniel graph containing a hole or domino $C$. Then $G$ has a skeleton $\Gamma$ (with respect to $C$) such that, for every vertex $z$ in $\Gamma$, the set $C_\Gamma(z)$ is not connected to $\Gamma_D$ in $G - D_\Gamma(z) - T$.

**Proof.** Let $\Gamma^0 = C$. By induction on $i$, we will prove that we can build a sequence of skeletons $\{\Gamma^i, i = 0, 1, \ldots, r\}$ of $G$ such that every of them satisfies the
two auxiliary properties
  A) If \( C' \) is an induced cycle of \( \Gamma^i \) then \( T_{C'} \subseteq T \).
  B) If \( w \in V \) and \( z_1, z_2 \in V(\Gamma^i) \) such that \( z_1 z_2 \in E, wz_1 \in E, wz_2 \in E \), then \( w \in D_{\Gamma^i}(z_1) \cup D_{\Gamma^i}(z_2) \cup T \).

and such that \( \Gamma_r \) is the skeleton we are looking for.

Remark 2 Notice that because of the property (2) of a skeleton of \( G \), every vertex \( z' \in D_{\Gamma'}(z) \) could be seen as the representative vertex of \( D_{\Gamma'}(z) \) in \( \Gamma^i \) instead of \( z \). Therefore the properties A), B) here above will hold for any of the copies of \( \Gamma^i \) obtained by considering vertices in each \( D_{\Gamma'}(z) \).

We will firstly prove that \( \Gamma^0 \) is a skeleton satisfying A), B), then, supposing that \( \Gamma^i \) is built and satisfies the hypothesis, we will build \( \Gamma^{i+1} \) and show it also verifies the hypothesis.

The statements A) and B) are easy to prove for \( \Gamma^0 \), using Claim 3. Let us show that \( \Gamma^0 \) is a skeleton of \( G \). To this end, we give for (2) and (3) proofs as general as possible (which do not depend on the bipartite graph \( \Gamma^0 \), but only on some of its properties; see the remarks below). Exactly the same proofs will be used later to show (2) and (3) for \( \Gamma^{i+1} \).

- There is nothing to do to prove (1).
- To prove (2), consider \( z, z' \in V(\Gamma^0) \) and suppose by contradiction that there exists \( z'' \in D_{\Gamma^0}(z') \) adjacent to \( z_1 \in D_{\Gamma^0}(z) \) but non-adjacent to \( z_2 \) in the same set. Since the graph induced by \( D_{\Gamma^0}(z) \) is connected, we may suppose that \( z_1 z_2 \in E \).

  We will prove that \( N_{\Gamma^0}(z) = N_{\Gamma^0}(z') \). To this end let, if it exists, \( u \) be a neighbour of \( z' \) on \( \Gamma^0 \) such that \( zu \not\in E \), and consider \( C' \) a smallest cycle of \( \Gamma^0 \) containing \( z'u \) and \( z \). Then \( C' \) contains at least one subpath \( U \) with extremities \( z \) and \( z' \), containing \( u \). Then \( U - \{z, z'\} + \{z_1, z_2, z''\} \) is a cycle whose chords but one have an extremity \( z'' \) (otherwise \( C' \) was not a smallest cycle). It is easy now to see that \( z'' \) must be adjacent to every vertex on \( U - \{z, z'\} \), and the same is valid for \( z' \) (since \( U \) is included in \( \Gamma^0 \) and \( z', z'' \) have the same behaviour with respect to the vertices in \( \Gamma^0 \)). We deduce that \( \Gamma^0 \) is not bipartite, a contradiction.

  A similar argument proves the other inclusion, so \( N_{\Gamma^0}(z') = N_{\Gamma^0}(z) \), therefore by the definition of the set \( D_{\Gamma}(w) \) (i.e. the largest connected set containing copies of \( w \)), \( z, z' \) should have been in the same set \( D_{\Gamma^0}(w) \), with \( w \in V(\Gamma^0) \) (they have the same neighbourhood in \( \Gamma^0 \) and are in the same connected component in the graph of copies, thanks to the edge \( z'' z_1 \)), a contradiction. So (2) is proved.

Remark 3 Notice that the proof of (2) does not depend on the graph \( \Gamma^0 \) that we use, but only on the fact that \( \Gamma^0 \) is bipartite and 2-connected.

- To prove (3), suppose the contrary holds for \( z \in V(\Gamma^0) \) and let \( P \) be a shortest path joining \( W_{\Gamma^0}(z) \) to \( \Gamma^0_D \) in \( G - D_{\Gamma^0}(z) - T \) (denote respectively by \( w, z_0 \) its extremities). Notice that the only vertex on this path which could have other neighbours on \( \Gamma^0_D - D_{\Gamma^0}(z) \) is the one preceding \( z_0 \), that we call \( s \) (otherwise \( P \) is not a shortest path). Since \( w \in W_{\Gamma^0}(z) \), there must exist two adjacent vertices \( z', z'' \in D_{\Gamma^0}(z) \) such that \( w z' \in E, w z'' \not\in E \). The graph induced by \( D_{\Gamma^0}(z) \) is
connected, so we can choose \( z', z'' \) such that \( z'z'' \in E \). Moreover, since (2) is proved we can (see Remark 2) suppose that \( z = z' \), by changing the representative vertex for \( D_{\Gamma^0}(z) \) if necessary.

Take \( P' \) a chordless path in \( \Gamma_D^0 \) from \( z_0 \) to \( z \), and denote \( z_1 \) the vertex which precedes \( z \) on this path. Without loss of generality, we can suppose that \( s \) has no neighbour on \( P' - \{ z_0, z \} \) (otherwise we suitably change \( z_0 \)); therefore in \( P + P' \), the only possible chords have an extremity \( z \) and the other one \( r \in V(P) \) such that \( r \in C_{\Gamma^0}(z) \) (otherwise \( P \) is not a shortest path, as defined). Consider (if it exists) \( r \neq w \) the neighbour of \( z \) the closest to \( w \) on \( P - w \). Because of \( zr \in E \) and \( r \in C_{\Gamma^0}(z) \), we also have \( z''r \in E \) and therefore \( rw \in E \) (otherwise \( z, z'' \), \( P_{wr} \) induce a bad cycle, since \( z'' \) cannot have other neighbours on \( P_{rw} \); they would be in \( W_{\Gamma^0}(z) \) and \( P \) wouldn't be as short as possible). Then \( z \) must also be adjacent to all the vertices on \( P + P' \) (apply Claim 1 for \( z \) with the neighbours \( z_1 \) and the appropriate vertex among \( r, w \), depending on the parity of the path), consequently \( z_0 = z_1 \). Now, the statement B) for \( s, z_0, z \) implies \( s \in D_{\Gamma^0}(z_0) \cup D_{\Gamma^0}(z) \cup T \), a contradiction.

We deduce that \( z \) has no neighbour but \( w \) (and possibly \( z_0 \) if \( z_0 = z_1 \)) on \( P \). Then \( P + P' \) is an induced cycle (on three or more vertices) and \( P + P' + z'' \) contradicts the hypothesis that \( G \) is a Meyniel graph, except if \( P + P' \) has exactly three vertices, i.e. \( s = w, z_0 = z_1 \). But then the theorem B) for \( s, z_0, z \) and \( \Gamma^0 \) gives another contradiction. Thus (3) is proved.

Remark 4 Notice that the proof of (3) does not depend on the graph \( \Gamma^0 \) that we use for \( G \), but only on the fact that \( \Gamma \) and (2) are true.

Now, we have that \( \Gamma^0 \) is a skeleton which satisfies A) and B). By induction, suppose that \( G \) has a skeleton \( \Gamma^i \) which satisfies A) and B). If no set \( C_{\Gamma^i}(z) \) is connected to \( \Gamma^i_D \) in \( G - D_{\Gamma^i}(z) - T \), then we denote \( \Gamma = \Gamma^i \) and Lemma 2 is proved. Otherwise, we build another skeleton \( \Gamma^{i+1} \) strictly containing \( \Gamma^i \) and also satisfying A) and B).

Since there exists some \( z \in V(\Gamma^i) \) such that \( C_{\Gamma^i}(z) \) is connected to \( \Gamma^i_D \) in \( G - D_{\Gamma^i}(z) - T \), let \( P \) be a shortest path joining a vertex of \( C_{\Gamma^i}(z) \) to \( \Gamma^i_D \) in \( G - D_{\Gamma^i}(z) - T \) (denote its extremities respectively by \( c \) and \( z_0 \)). Again, only \( s \) (the vertex preceding \( z_0 \) on \( P \)) can have other neighbours on \( \Gamma^i_D - D_{\Gamma^i}(z) \) (because of the minimality of \( P \)). Once more, take \( P' \) a chordless path joining \( z_0 \) and \( z \) in \( \Gamma^i_D \) and suppose that \( s \) has no neighbour on \( P' - \{ z_0, z \} \). Notice that \( P + P' \) has no chord; otherwise, such a chord would have the extremities \( z, r \) with \( r \in V(P) \) and \( r \) could be neither in \( W_{\Gamma^i}(z) \) (since \( W_{\Gamma^i}(z) \) is not connected to \( \Gamma^i_D \)) nor in \( C_{\Gamma^i}(z) \) (the path \( P \) wouldn't be the shortest with the indicated properties).

Define \( \Gamma^{i+1} = \Gamma^i + P \) (obviously, this is a bipartite 2-connected graph) and let us prove that \( \Gamma^{i+1} \) is a skeleton of \( G \) satisfying A) and B).

A) is true for \( \Gamma^{i+1} \)

Let \( C' \) be an induced cycle of \( \Gamma^{i+1} \). If it is also a subgraph of \( \Gamma^i \), then by induction we have the desired conclusion. Otherwise, \( V(C') \cap V(P) \neq \emptyset \) implies that \( s \in V(C') \). As \( s \) can have more than one neighbour on \( \Gamma^i \), two possibilities
may appear:

**First case.** The neighbours of \( s \) on \( C' \) are \( z_1, z_2 \in V(\Gamma^i) \).

Let \( t_1 \in T_{C'} \). If the path (included in \( \Gamma^i \)) \( C' - \{ z_1, z_2, s \} \) has more than one vertex, then by applying B) in \( \Gamma^i \) for \( t_1 \) and two adjacent vertices on this path we deduce that \( t_1 \in T \) (the other two possibilities are not valid, since \( t_1 \) has two many neighbours on \( \Gamma^i \)). If this path has exactly one vertex \( z_3 \), then by applying B) for \( \Gamma^i \) and \( t_1, z_1, z_3 \) we deduce that \( t_1 \in D_{\Gamma^i}(z_3) \cup T \). If \( t_1 \in T \) then we are done; in the other case, \( s \) is adjacent to \( z_1, t_1 \) (for which the property B) may be applied, thanks to Remark 2) and we have \( s \in D_{\Gamma^i}(z_1) \cup D_{\Gamma^i}(z_3) \cup T \), a contradiction.

**Second case.** Only one neighbour of \( s \) on \( C' \) is in \( \Gamma^i \).

Then \( C' \) contains the whole path \( P_{s\bar{s}} \), and the rest of \( C' \) is a path in \( \Gamma^i \) with an extremity \( z \) (the neighbour of \( c \) in \( \Gamma^i \)) and the other one denoted \( z_1 \). If this path has more than three vertices, then by B) for \( \Gamma^i \) and any \( t_1 \in T_{C'} \) we have \( t_1 \in T \). If it has exactly three vertices (say \( z_1, z_2, z \)), then by B) we have \( t_1 \in D_{\Gamma^i}(z_2) \cup T \), but \( t_1 \in D_{\Gamma^i}(z_2) \) is impossible since then \( c, t_1, z \) would give by B) a contradiction. If it has exactly two vertices \( z_1 \) and \( z \), then again by B) we must have \( t_1 \in D_{\Gamma^i}(z_1) \cup D_{\Gamma^i}(z) \cup T \). The only possible case is \( t_1 \in T \) (the other two imply that either \( c \) or \( s \) has two adjacent neighbours on \( \Gamma^i \)).

Then A) is proved for \( \Gamma^{i+1} \).

**B) is true for \( \Gamma^{i+1} \)**

Take \( w, z_1, z_2 \) as indicated. We will prove that if \( w \notin T \), then \( w \in D_{\Gamma^{i+1}}(z_1) \cup D_{\Gamma^{i+1}}(z_2) \). To this end, take \( H_1 = [z_1, z_2, z_3, \ldots, z_k, z_1] \) a chordless cycle of \( \Gamma^{i+1} \) containing \( z_1, z_2 \). Claim 3 for this cycle implies that either \( w \) is adjacent exclusively to \( z_k \) on \( H_1 \) \( - \{ z_1, z_2 \} \) or exclusively to \( z_3 \) (\( w \notin T_{H_1} \) otherwise because of the statement A) for \( \Gamma^{i+1} \), we should have \( w \in T \)). Without loss of generality, suppose that the first possibility is valid (in the second one, the reasoning is similar) and let us prove that \( w \in D_{\Gamma^{i+1}}(z_1) \), that is \( N_{\Gamma^{i+1}}(w) - \{ z_1 \} = N_{\Gamma^{i+1}}(z_1) \).

We prove \( w \in D_{\Gamma^{i+1}}(z_1) \).

By contradiction, suppose firstly that there exists a neighbour \( z' \in V(\Gamma^{i+1}) \) of \( w \), which is not a neighbour of \( z_1 \). Take \( H_2 = [z_1, z_2, u_1, \ldots, u_p, z_1] \) a shortest cycle of \( \Gamma^{i+1} \) containing \( z_1, z_2 \) and \( z' \), and denote \( U \) a chordless subpath of the graph induced by \( z_1, z_2, u_1, \ldots, u_p \) joining \( z_1 \) and \( u_1 = z' \) (Notice that \( u_1, \ldots, u_p \) is chordless and all the chords of the path \( [z_1, z_2, u_1, \ldots, u_p] \) contain \( z_1 \)).

- if \( z_2 \notin V(U) \) (i.e. \( z_1 \) has a neighbour on \( [u_2, \ldots, u_p] \)), then Claim 1 for \( w \) and \( U \), or for \( w \) and \( U + z_2 \) (depending on the parity) implies that \( w \) is adjacent to every vertex on \( U \). Since \( z_1 z' \notin E \), \( U + z_2 \) has at least four vertices, so \( w \) is adjacent to a \( P_4 \) of \( \Gamma^{i+1} \) (the graph on four vertices cannot be a \( C_4 \) since by A) we would obtain \( w \notin T \)). Let us show that in the second case (see here below) we have the same conclusion.

- if \( z_2 \in V(U) \), then Claim 1 for \( w \) and \( U \), or for \( w \) and \( U - z_1 \) (depending on the parity) implies again that \( w \) is adjacent to every vertex on \( U \). Now, the case where \( z' \) is not the other neighbour of \( z_2 \) on \( U \) easily implies that \( w \) is adjacent to every vertex of a \( P_4 \) of \( \Gamma^{i+1} \). The contrary case, i.e. \( z' z_2 \in E \), gives that \( z_k \) (see the reasoning above), \( z_1, z_2, z' \) form another \( P_4 \) whose vertices are adjacent to \( w \).
We have to prove now that if \( w \) is adjacent to a \( P_4 \) \( abcd \) of \( \Gamma^{i+1} \), it must be adjacent to the whole \( \Gamma^{i+1} \). Let \( H \) be a shortest cycle in \( \Gamma^{i+1} \) containing \( ab \) and \( cd 
\).

- If the four vertices are in the order \( a, b, c, d \) on \( H \), then notice that \( bc \) is an edge of the cycle and the only possible chords have an extremity \( b \) or \( c \) (otherwise \( H \) is not as short as possible). Since \( H \) is an even cycle, \( H - \{b, c\} \) is an odd induced path and implies, with \( w \) and by Claim 1, that \( w \) is adjacent to every vertex in \( H \). By the statement \( A \) for a chordless subcycle of \( H \) we have the desired conclusion.

- If the four vertices are in the order \( a, c, d, b \) on \( H \), then call \( U_1 \) the subpath of \( H \) joining \( a, c \) not containing \( d \), and \( U_2 \) the subpath of \( H \) joining \( b, d \) not containing \( a \). It is clear that the paths \( U_1 \) and \( U_2 \) have no chord (otherwise \( H \) is not as short as possible). Moreover, if there exists a chord \( xy \) on \( H \) joining a vertex from \( U_1 - c \) to a vertex from \( U_2 - b \) then we can found a cycle \( H' \) containing \( xy \) and such that \( a, b, c, d \) are in this order on \( H' \) (\( H' \) must contain all the vertices of \( H \), since \( H \) is as short as possible; moreover, for \( H' \) we can perform the preceding reasoning to obtain the desired conclusion). Thus, we may suppose that all the chords of \( H \) have an extremity \( b \) or \( c \) and the other one respectively on \( U_1 - a \) or \( U_2 - d \). Therefore, \( w, a \) and \( U_2 \) imply, by Claim 1, that \( w \) is adjacent to all the vertices on \( U_2 \). Then \( w \) is adjacent to all the vertices of a chordless subcycle of \( U_2 + c \) and by \( A \) we have \( w \in T \), a contradiction.

We prove "\( z' \)."

Suppose now that there exists a neighbour \( z' \in V(\Gamma^{i+1}) \) of \( z_1 \), which is not a neighbour of \( w \). Then there exists in \( \Gamma^{i+1} \) a cycle \( H \) containing \( z_1 z' \) and \( z_1 z_2 \) whose only chords have an extremity \( z_1 \). Call \( U = H - z_1 \). Among all the neighbours of \( z_1 \) on \( U \) (\( z' \) and \( z_2 \) included), there exists at least one (namely \( z' \)) which is non-adjacent to \( w \). Consider such a neighbour \( z_3 \), which is the closest to \( z_2 \) along \( U \) and let \( z_4 \) be the neighbour of \( z_1 \) on \( Uz_3z_2 - z_3 \) the closest to \( z_3 \). Then \( z_4w \in E \), \( z_3w \notin E \). The cycle induced by \( z, z_1, Uz_3z_4 \) is odd and has only one chord (because of \( N_{\Gamma^{i+1}}(w) \subseteq N_{\Gamma^{i+1}}(z_1) \), \( w \) cannot have neighbours on \( Uz_3z_4 - z_4 \), a contradiction.

The statement \( B \) is proved.

Now, to show that \( \Gamma^{i+1} \) is a skeleton of \( G \), we should prove the conditions (1), (2), (3) hold. But (1) is obviously true, and (2), (3) immediately follow from Remarks 3, 4.

Lemma 2 is proved: as long as the indicated condition on the sets \( C_{\Gamma}(z) \) is not verified, a new graph \( \Gamma^{i+1} \) may be built, with the same properties. When it is verified, the last graph we have built is the one we were looking for (call it \( \Gamma \)). □

If we call \( I_\Gamma(z) \) the set \( W_\Gamma(z) \cup C_{\Gamma}(z) \cup L_\Gamma(z) \), for any \( z \in \Gamma \), then we have the announced property, i.e. \( I_\Gamma(z) \) may be consider as “attached”, in \( G - T - S \), to \( D_\Gamma(z) \) since it has no other connections to the rest of \( G - T - S \).

Remark 5 There is no difficulty to see that every vertex \( t \in T \) is adjacent, in fact, to every vertex in \( \cup_{z \in C} D_\Gamma(z) \). Indeed, take arbitrary vertices \( z \in C, z' \in D_\Gamma(z) \) and simply apply Claim 3 for \( t \) and the graph \( C - z + z' \) (isomorphic to \( C \)).
On the contrary, while considering other vertices $z$ (of $\Gamma - C$), $T$ may be adjacent to only a part of $D_T(z)$. This why we will try to improve this structure in order to have an homogeneous behaviour of $T$ with respect to the skeleton.

4. Improving the skeleton

Let us limit $\Gamma$ to its induced subgraph $\Gamma^i$ containing all the vertices $z \in \Gamma$ such that every vertex of $D_T(z)$ is adjacent to every vertex of $T$. Obviously, $\Gamma^i$ is a bipartite graph containing $C$. Moreover, we have:

**Lemma 3** The graph $\Gamma^i$ is 2-connected.

**Proof.** If this is not the case, then consider $K$ the 2-connected component of $\Gamma^i$ containing $C$, and $z \in V(K)$ a vertex such that $\Gamma^i - z$ is not connected. Therefore, there exists a 2-connected component $R$ of $\Gamma^i$ which contains $z$ and has no other connections with $K$.

Since $\Gamma$ is 2-connected, there must exist a path joining $R$ and $K$ in $\Gamma - z$. Consider $U$ a shortest path joining in $\Gamma - z$ a vertex in $R$ to a vertex in $K$; denote $u \in V(R)$ and $v \in V(K)$ its extremities, $I$ a chordless path joining $z, u$ in $R$ and $J$ a chordless path joining $z, v$ in $K$. Without loss of generality, we may suppose that no chord exists joining vertices from $I - z$ or $J - z$ to vertices of $U$. Then the only possible chords of $U + I + J$ have an extremity $z$ and the other one on $U$.

**Claim 4** If $r$ is the neighbour of $z$ on $I$ and $w \in V(K)$ is a neighbour of $v$ out of $J$, then $r = u$ and $vz, wu \in E$.

**Proof.** It is quite easy to see that the path $I + J$ contains at least three vertices, and $U - R - K$ at least one (situated in $\Gamma - \Gamma^i$). We have two possibilities:

- If the path $I + J$ contains at least four vertices, then $I$ or $J$ contains more that two vertices. We suppose that $I$ has at least three vertices (otherwise, by replacing $I$ with $J$ and by a similar proof, we obtain the same conclusion). By applying Claim 1 for any $t \in T$ and the path $I + U + J - \{r, z\}$ we obtain that $t$ is adjacent to any vertex on $U + I + J$.

Now, since any vertex $q$ on $U + I + J$ is situated on an induced subgraph of $U + I + J$ isomorphic to a hole or a domino (this is a consequence of the fact that $\Gamma$ is bipartite and all the chords in $U + I + J$ have an extremity $z$), as in Remark 5 we deduce that any $t \in T$ is adjacent to any vertex in $D_T(q)$, for an arbitrary $q \in U$. But then every vertex in $U$ should be in $\Gamma^i$, a contradiction.

- If the path $I + J$ contains exactly three vertices, then both $I, J$ are paths of length one, whose vertices will be $z, u$ (in this case $u = r$), respectively $z, v$.

Take $w \neq z$ a neighbour of $v$ on $K$ and suppose by contradiction that $wu \notin E$. Let $q$ be the neighbour of $v$ on $U$. We can see that every vertex of $U - v$ is non-adjacent to $w$ (this is true for $q$ because $\Gamma$ is bipartite and for each vertex of $U - \{q, v\}$ because $U$ is as short as possible). Then, we apply Claim 1 for any $t \in T$ and the path $U + w$ to deduce that $t$ is adjacent to all the vertices of $U$.

Now, we deduce as above that $V(U) \subseteq V(\Gamma^i)$ (a contradiction), if $U + z$ has at least six vertices. On the contrary, if $U + z$ has exactly four vertices, the same
reasoning doesn’t work: but if \( q' \in D_T(q) \) (where \( q \) is the unique vertex on \( U-I-J \)),
the cycle given by \( t, u, q', v, w \) implies \( tq' \in E \). Claim 4 is proved. \( \square \)

We can easily conclude the proof of Lemma 3: the Claim before implies that \( u \) and \( K \) are, in fact, in the same 2-connected component of \( \Gamma^l \), a contradiction. \( \square \)

So \( \Gamma^l \) is bipartite and 2-connected; but it is not necessarily a skeleton of \( G \)
satisfying Lemma 2, since for two vertices \( z_1, z_2 \in V(\Gamma^l) \) the sets \( C_{\Gamma^l}(z_1), C_{\Gamma^l}(z_2) \)
may be connected in \( G-G' \) if \( N_{\Gamma^l}(z_1) = N_{\Gamma^l}(z_2) \) (see below). But then \( z_1, z_2 \) may
be seen as “copies” of the same vertex, since their role in the graph \( \Gamma^l \) is the same;
in other words, they will be a part of a module of \( \Gamma^l \).

As recalled in Section 1, the modular decomposition associates to any graph
a representative graph, which is either a clique, or a stable set, or a prime graph
(i.e. a graph with no modules) such that the initial graph may be obtained from
the representative graph by replacing any of its vertices with a module. Apply the
modular decomposition to our graph \( \Gamma^l \), call \( \Gamma' \) the associated representative graph
(which is a prime graph) and notice that \( \Gamma' \) necessarily contains \( C \). Indeed, if this
is not the case, then \( C \) should be in a module (\( C \) is prime, so all its vertices will
always be in the same module). Since \( \Gamma' \) is connected, all the vertices of this module
should be adjacent to all the vertices of another module. But then \( \Gamma^l \) wouldn’t be
bipartite.

So \( \Gamma' \) contains \( C \) and \( \Gamma^l \) is obtained from \( \Gamma' \) by replacing any vertex with a
set of vertices (which, in fact, should induce stable sets, otherwise \( \Gamma^l \) wouldn’t be
bipartite). Obviously, \( \Gamma' \) is bipartite; but it is not necessarily 2-connected. Then
let \( \Gamma' \) be the 2-connected component of \( \Gamma' \) containing \( C \), and \( z \) an arbitrary vertex
of \( \Gamma' \) representing, as a vertex of \( \Gamma' \), the vertices \( z_1, z_2, \ldots, z_p \) of \( \Gamma^l \). We denote:

- \( D(z) = \cup_{i=1}^p D_{\Gamma'}(z_i) \), where, in fact, \( D_{\Gamma'}(z_i) = D_{\Gamma}(z_i) \)
- \( \Gamma'_D \) the subgraph induced by the vertices in \( \cup_{i \in V(\Gamma')} D(z) \)
- \( W(z) = \{ w \in V - V(\Gamma'_D) - T \mid w \text{ is adjacent to some vertices in } D(z) \} \)
- \( L(z) = \{ w \in V - W(z) - D(z) - T \mid w \text{ is connected to } W(z) \text{ in } G - D(z) - T \} \).

As it can be easily seen, these notations closely follow the initial ones, the only two
differences being the following ones:

1. the sets \( D(z) \) are not necessarily connected, while \( D_{\Gamma}(z) \) were;
2. the vertices \( w \in V - V(\Gamma'_D) - T \) adjacent to some vertices in \( D(z) \) are no
   more separated in \( W(z) \) and \( C(z) \), but put together in the set \( W(z) \). Indeed, as
   long as these vertices do not have different behaviours with respect to the rest of
   the graph, there is no reason to treat them differently. This was necessary to be
done in the inductive processing for the skeletons \( \Gamma^l \), but it is no more necessary.

Now, we prove that \( \Gamma' \) is a proper skeleton of \( G \) (with respect to \( C \)), i.e.:

1. \( \Gamma' \) is bipartite 2-connected and contains \( C \);
2. for every \( z \in V(\Gamma') \), \( D(z) \) is a module in \( \Gamma'_D \);
3. for every \( z \in V(\Gamma') \), \( W(z) \) is not connected to \( \Gamma'_D \) in \( G - D(z) - T \);
4. every vertex in \( \Gamma'_D \) is adjacent to all the vertices of \( T \).

Once more, these properties are similar to the one of a skeleton, except for the
fourth one.
Lemma 4 Let $G$ be a Meyniel graph containing an induced hole or domino $C$. Then the graph $\Gamma'$ built as described is a proper skeleton of $G$ (with respect to $C$).

Proof. Properties (1') and (4') are obvious. Property (2') follows immediately from the construction of $\Gamma'$ and property (2) of $\Gamma$. We only have to prove (3').

By contradiction, suppose there exists a vertex $z \in V(\Gamma')$ such that $W(z)$ is connected to $\Gamma'_D$ in $G - D(z) - T$. Take $U$ a shortest path joining $W(z)$ to $\Gamma'_D$ in $G - D(z) - T$. Suppose that $W$ were not a different set of generality that the vertex $v$ preceding $U$ has no other neighbour on $J$. Then the cycle $U + J$ has no chords (otherwise $U$ wouldn't be as short as possible).

Remark 6 The path $U$ contains at least one vertex $q$ of $\Gamma - \Gamma'$ (i.e. such that not every $t \in T$ is adjacent to all $q' \in D_t(q)$), otherwise $U$ should be included in $\Gamma'$.

Consider $q$ chosen as close as possible to $r$.

Two possibilities exist for $r$.

- If $r \in V(\Gamma')$, then denote $u$ the vertex preceding $q$ on $U_q$. Let $w$ be a neighbour of $v$ on $K - J$ we may apply Claim 4 to deduce that $wr \in E$, so $r$ is in the same 2-connected component of $\Gamma'_D$ as $\Gamma'$, a contradiction.

- If $r \notin V(\Gamma')$, then there exists $q' \in D_t(v)$ and $t \in T$ such that $tr' \notin E$. Denote $U' = U - r + r'$ (clearly, $tr' + J$ has no chord).

First case: $J$ has at least four vertices. Then by Claim 3 we have that $t$ is adjacent to all the cycle $U' + J$, and this contradicts the assumption $tr' \notin E$.

Second case: $J$ has exactly three vertices. Then in both cases $|V(U' + J)| \geq 6$ and $|V(U' + J)| = 4$ we can deduce that $N_{\Gamma'}(z) = N_{\Gamma'}(v)$. To this end, let us firstly show that in both cases we can assume that $t$ has no neighbour on $U' - v$.

If $|V(U' + J)| \geq 6$, then the contrary should imply by Claim 3 that $tr' \notin E$, a contradiction. If $|V(U' + J)| = 4$, then $V(U') \setminus \{v\} = \{r'\}$ and $tr' \notin E$.

In both cases we prove now that $N_{\Gamma'}(v) = N_{\Gamma'}(z)$. Indeed, if $w \in N_{\Gamma'}(v) - N_{\Gamma'}(z)$, then Claim 3 for the cycle given by $t, z, U'$ and the vertex $w$ implies (since $\Gamma$ is bipartite) that $wz \in E$ (notice that $w$ is not in $U'$ since $t$ has no other neighbour on $U' - v$, as proved before). Conversely, if $w \in N_{\Gamma'}(z) - N_{\Gamma'}(v)$, then again by Claim 3 for the cycle given by $t, z, U'$ and the vertex $w$ implies (again, since $\Gamma$ is bipartite) $uv \in E$. So the neighbourhoods of $v$ and $z$ in $\Gamma'_D$ are identical.

Since $\Gamma'_D$ is a prime graph, this is not possible.

Third case: $J$ has exactly two vertices. Then (recall that $s$ is the neighbour of $v$ on $U'$) Claim 3 for any cycle $U' + J - s + s'$ with $s' \in D_t(s)$ and any $t \in T$, implies that $ts' \in E$ so $s \in V(\Gamma')$. Now, if we still call $J$ the path $J + s$, $U'$ the path $U' - s$ and we apply the reasoning in the case before (where $J$ has exactly three vertices),
we obtain that \(N_{\Gamma'}(s) = N_{\Gamma'}(z)\). This is a contradiction with the hypothesis on \(\Gamma'\) (\(s\) should be in the module of \(\Gamma'\) represented by \(z\) in \(\Gamma'_1\)), so (3') is proved. \(\square\)

This method for finding \(\Gamma'\), with its three steps (building \(\Gamma', \Gamma'_1, \Gamma'\)), will be called the refining operation.

5. Characterization Theorem

One more simple structural result is necessary before proving the main theorem. Obviously, the statement A), which was true for any of the graphs \(\Gamma'\) in the inductive process (see Lemma 2), is also true in \(\Gamma'\). The statement B) and a new one are proved here below:

**Claim 5** The following two statements hold for \(G\) with the proper skeleton \(\Gamma'\):

B) If \(w \in V\) and \(z_1, z_2 \in V(\Gamma')\) such that \(z_1z_2 \in E, wz_1 \in E, wz_2 \in E\), then \(w \in D(z_1) \cup D(z_2) \cup T\).

C) If \(t_1, t_2 \in T\) such that \(t_1t_2 \notin E\), then \(N_G \ T \ s(t_1) = N_G \ T \ s(t_2)\).

**Proof. B) is true.**

Using the statement B) for \(\Gamma\), we obtain that \(w \in D_{\Gamma}(z_1) \cup D_{\Gamma}(z_2) \cup T\). Since \(D_{\Gamma}(z_1) \subseteq D(z_1)\) and \(D_{\Gamma}(z_2) \subseteq D(z_2)\) (by definition), we are done.

C) is true.

Obviously, if this is not the case there must exist \(z \in V(\Gamma')\) and \(r \in W(z) \cup L(z)\) such that \(rt_1 \in E\) but \(rt_2 \notin E\). Take \(U\) a chordless path from \(r\) to \(z\) in the subgraph induced by \(D(z) \cup W(z) \cup L(z)\) (if it doesn’t exist, since \(D(z)\) could be non-connected, we suitably change the representative vertex \(z\) of the set \(D(z)\) in \(\Gamma'\)) and suppose, without loss of generality, that \(r\) is the first vertex on this path, while going from \(z\) to \(r\), such that \(r\) is adjacent to exactly one of the two vertices \(t_1, t_2\). Suppose also that \(z\) is the only vertex of \(D(z)\) on \(U\) (otherwise we can again change the representative vertex of \(D(z)\) on \(\Gamma'\)). Let \(z'\) be a neighbour of \(z\) on \(\Gamma'\) and notice that both \(z\) and \(z'\) are adjacent to \(t_1, t_2\). Then take the cycle induced by \(z', t_1, U\). If it is odd, then Claim 1 for \(t_1\) and \(U + z'\) implies that \(t_1\) is adjacent to every vertex on \(U\); if it is even, the same result is obtained by applying Claim 1 for \(t_1\) and \(U\). In both cases we must have, by the choice of \(U\), that \(t_2\) is adjacent to any vertex on \(U\), except \(r\). Now, take \(s\) the neighbour of \(r\) on \(U\) and \(z'' \in V(\Gamma')\) such that \(z''z \notin E\). Then the cycle given by \(r, s, t_1, t_2, z''\) is odd and has only one chord, a contradiction (by (3') for \(\Gamma', r, z'' \notin E\)). \(\square\)

While performing the preceding reasoning, the condition for \(G\) to be a Meyniel graph has been used locally (around the skeleton \(\Gamma'\)), but not globally (parts of \(W(z) \cup L(z)\) may have not been tested). Therefore, it is not sufficient for a graph to have a proper skeleton to deduce that it is a Meyniel graph. However, the graph may be reduced, by removing some edges of \(\Gamma'_D\), to a smaller (with respect to the number of edges) graph which has the same nature as \(G\) (Meyniel graph or not) but a smaller number of holes and dominoes. This is done as follows.

Consider \(S_{\Gamma'}\) a spanning tree of \(\Gamma'\) and let \(G'\) be the graph obtained from \(G\) by removing all the edges in \(\Gamma'_D\) between two sets \(D(z_1), D(z_2)\) with \(z_1, z_2 \in V(\Gamma')\).
In other words, the structure identified around $\Gamma'$ remains unchanged, but $\Gamma'$ is reduced to a spanning tree by removing some of its edges. Nothing is changed inside the sets $D(z), W(z), L(z), T, S$ (for any $z \in V(\Gamma')$). Notice that in this way $C$ is destroyed (and possibly some other holes or dominoes of $G$). It remains to prove that the nature of the graph doesn’t change.

**Theorem 1** *(Characterization Theorem)* The graph $G = (V, E)$ is a Meyniel graph if and only if either it is HHD-free or, for any even hole or domino $C$, the conditions below are simultaneously verified:

1. the graph $\Gamma'$ (obtained by the refining operation) is a proper skeleton of $G$;
2. the reduced graph $G'$ (with respect to any spanning tree $S_{T'}$) is a Meyniel graph.

**Proof.** "⇒": The proof of 1. has already been done, so we will prove 2. To this end, suppose the contrary holds and take $C' = [v_1, v_2, \ldots, v_k, v_1]$ (for odd, at least equal to 5) a cycle of $G'$ with at most one chord. Now, since $G$ is a Meyniel graph, at least one chord of the cycle induced by $V(C')$ exists in $G$ but is removed in $G'$. Say this chord is $v_i v_k$ and suppose without loss of generality that $i > 3$ (the case $v_i v_k \in E$ is similar to the case $v_i v_{k-1} \in E$). Hence, $v_i v_k$ is an edge of $\Gamma'_D$ and we can, without loss of generality, suppose that $v_i, v_k \in V(\Gamma')$. Let us distinguish two cases:

- There exists $x \in V(C'_{v_i v_{i+1}})$ such that $x \in T \cup S$.

  If $x \in T$, then we have $x = v_2$ or $x = v_{i-1}$ (otherwise $xv_i$ and $xv_j$ are two chords of $C'$ in $G'$). Suppose without loss of generality that $x = v_2$ (so $v_2 v_i$ is a chord) and let $y \in V(C'_{v_{i+1}, v_k})$. If $y \in V(\Gamma'_D)$, then $yv_2$ is the second chord of $C'$ and we have a contradiction. Suppose that $y \in T$. Then $yv_2 \notin E$ (this would be a second chord in $C'$) so, by the property C, $N_G(T \setminus s(v_2)) = N_G(T \setminus s(y))$; we deduce that $v_3$ is not in $G - T - S$ ($v_3 y$ would be the second chord of $C'$ in $G'$), therefore $v_3 \in T$ (if $v_3 \in S$, then another vertex on $C'_{v_i v_{i+1}}$ would be in $T$, and another chord will be found). But now $v_3 v_1 \in E$ and we have again a contradiction. Since $y \notin T$, we also have $y \notin S$ (otherwise there necessarily exists a vertex of $C'_{v_i v_{i+1}}$ which belongs to $S$) and we can deduce that $C'_{v_i v_{i+1}}$ has all its vertices in $W(v_i) \cup W(v_k)$; but no vertex of $W(v_i)$ is adjacent to a vertex of $W(v_k)$ in $G'$. This is a contradiction with the fact that $C'_{v_i v_{i+1}}$ is connected.

  If $x \in S$, then there exists some vertices of $C'_{v_i v_{i+1}}$ which belong to $T$, and we obtain, as previously a contradiction.

- $C'_{v_i v_{i+1}}$ is completely contained in $G' - T - S$.

  In both cases $V(C'_{v_i v_{i+1}}) \subseteq V(\Gamma'_D)$ and $V(C'_{v_i v_{i+1}}) \subseteq V(\Gamma'_D)$, we can find on $C'_{v_i v_{i+1}}$ at least two vertices of $\Gamma'_D$. In the first case this is obvious since $v_2 \neq v_{i-1}$. In the second one, at least one $z \in V(\Gamma')$ exists such that $V(C'_{v_i v_{i+1}}) \cap W(z) \neq \emptyset$. Then $|V(C'_{v_i v_{i+1}}) \cap D(z)| \geq 2$, since any path from $W(z)$ to $\Gamma'_D$ must have vertices from $D(z)$. If both these vertices are on $C'_{v_i v_{i+1}}$, then we are done; if, for instance, $v_i \in D(z)$, the second vertex $v_j \in D(z)$ ($j < i$) is not the neighbour of $v_i$ on $C'$ (since then $v_i v_j$ should have been deleted). We deduce that at least one vertex of $\Gamma'_D$ exists on $C'_{v_{i+1} v_{i+2}}$ ($v_i \notin D(z)$ since $v_1 \in D(z)$, so $V(C'_{v_{i+1} v_{i+2}}) \subseteq W(z)$).
Thus, $C'_{v_{a+1},v_{b}}$ is a subgraph of $G' - T - S$ (otherwise, there exists a vertex of $V(C'_{v_{a+1},v_{b}}) \cap T$ which forms two chords with two vertices of $V(C'_{v_{a+1},v_{b}}) \cap \Gamma'_D$, a contradiction). Therefore $C'$ is also a subgraph of $G' - T - S$. Notice that at least two vertices $v_a, v_b$ of $C'$ must be situated in the same set $\mathcal{D}(z)$. Indeed, if $V(C') \subseteq V(\Gamma'_D)$, then $C'$ (or a copy of it) cannot be induced exclusively by vertices of $\Gamma'$, since $\Gamma'$ is bipartite; the indicated conclusion follows. If $V(C') \not\subseteq V(\Gamma'_D)$, at least one vertex $z \in V(\Gamma')$ exists such that $V(C') \cap W(z) \neq \emptyset$. This implies as before that $|V(C') \cap \mathcal{D}(z)| \geq 2$.

We can suppose without loss of generality that $z \neq v_1$ and $a < b$. We choose $a$ as small as possible (so $v_{a+1} \in V(\Gamma'_D) - D(z)$), but take more precautions for choosing $b$. If $z \neq v_1$, then we necessarily can find $b < i$ such that $v_{b+1} \in V(\Gamma'_D) - D(z)$ (it is sufficient to take the largest $b$ smaller than $i$ such that $z_b \in D(z)$). If $z = v_1$, then we possibly cannot find such a $b$ smaller than $i$. Then we take $b$ as small as possible (larger than $i$) such that $v_{b+1} \in V(\Gamma'_D) - D(z)$ (if $b = k$ we denote $v_{b+1} = v_1$).

If $v_a, v_{a+1}$ are two chords of $C'$ in $G'$, we have a contradiction ($v_{a}, v_{a+1}, v_{b+1}$ are not removed in $G'$, since $v_{a}, v_{b+1}$ are not). Otherwise, that is if $v_a, v_{a+1} = v_1$, we necessarily have $v_1 = z$ and then $v_{a+1}, v_{b+1}$ should have been removed in $G'$, as well as $v_1$ (since $v_i, v_a, v_b \in D(z)$), a contradiction.

"\therefore" Suppose now that $\Gamma'$ was find by the refining operation and that $G'$ is a Meyniel graph, but $G$ is not a Meyniel graph. Once more, there must exists a cycle $C' = [v_1, \ldots, v_k, v_1]$ (k odd, at least equal to 5) of $G$ containing at most one chord. Since $G'$ is a Meyniel graph, $C'$ cannot induce a cycle in $G'$ (it would have at most one chord, too), therefore at least one edge of $C'$ (say $v_1 v_2$) is missing in $G'$.

Hence, $v_1 v_2$ corresponds to an edge $z_1 z_2$ of $\Gamma'$ not contained in $\mathcal{S}_{T'}$. There is no loss of generality if we suppose that $v_1, v_2 \in \Gamma'$. Two cases can occur:

- There exists $x \in V(C'_{v_1,v_2})$ such that $x \in T \cup S$.

  If $x \in T$, then we have $x = v_3$ or $x = v_k$ (otherwise $x v_1$ and $x v_2$ are two chords of $C'$ in $G$). Suppose without loss of generality that $x = v_3$ (so $v_1 v_3$ is a chord). Now, no vertex $y$ of $C'_{v_1,v_2}$ is in $T \cup V(\Gamma'_D)$ (otherwise $v_1 v_2$ and $y v_2$, respectively $y v_3$ are two chords of $C'$ in $G$). Then $V(C'_{v_1,v_2}) \subseteq W(v_1) \cup L(v_1) \cup S$. In fact, no vertex of $C'_{v_1,v_2}$ is in $S$ since in this case any path joining it to $v_1$ should contain a vertex in $T$. Hence $V(C'_{v_1,v_2}) \subseteq W(v_1) \cup L(v_1) \cup S$, and then $v_1 \in D(v_1) \cup T \cup L(v_1)$. In the first two cases $v_3 v_1$ and $v_1 v_2$ are two chords of $C'$ in $G$, a contradiction. In the third one, recall that there exists $y \in V(\Gamma')$ such that $y v_1 \in \mathcal{S}_{T'}$. Then $C' - v_2 + y$ is a cycle of $G'$ with only one chord, a contradiction.

  If $x \in S$, then there exists some vertices of $C'_{v_1,v_2}$ which belong to $T$, and we obtain, as previously, a contradiction.

- $C'_{v_1,v_2}$ is completely contained in $G - T - S$.

Once more, at least two vertices $v_a, v_b$ of $C'$ must be situated in the same set $\mathcal{D}(z)$. Indeed, if $V(C') \subseteq V(\Gamma'_D)$, then $C'$ (or a copy of it) cannot be induced exclusively by vertices of $\Gamma'$, since $\Gamma'$ is bipartite. If $V(C') \not\subseteq V(\Gamma'_D)$, at least one vertex $z \in V(\Gamma')$ exists such that $V(C') \cap W(z) \neq \emptyset$. This implies as before that $|V(C') \cap \mathcal{D}(z)| \geq 2$; call $z_a, z_b$ two vertices in the intersection.
We can suppose without loss of generality that \( z \neq v_1 \) (so \( v_a \neq v_1 \) and \( v_b \neq v_1 \)) and \( a < b \). We choose \( b \) as large as possible (if \( b = k \) we denote \( v_{b+1} = v_1 \), so \( v_{b+1} \in V(\Gamma_D') - D(z) \). We choose \( a \) the largest possible (smaller than \( b \)) such that \( v_a \not\in V(\Gamma_D') - D(z) \). If \( v_a \neq v_2 \) or \( v_b \neq v_b \), then we have a contradiction (\( v_a, v_{b+1}, v_b, v_a \) are two chords of \( C' \) in \( G \)). Otherwise, we have \( z = v_2 \) and \( V(C'_{v_2v_a}) \subseteq D(v_2) \cup W(v_2) \cup L(v_2) \). The edge \( v_1v_2 \) is not an edge of \( \Gamma' \), but \( v_2 \) has at least one neighbour \( y \in V(\Gamma') \) such that the edge \( v_2y \) is an edge of \( \Gamma' \) (otherwise \( \Gamma' \) is not a spanning tree). But then \( C' - v_1 + y \) is isomorphic to \( C' \) and is an induced subgraph of \( G' \), so \( G' \) is not a Meyniel graph. This contradicts the hypothesis. \( \square \)

6. The Algorithm

The characterization theorem gives us now the possibility to indicate a recognition algorithm for Meyniel graphs. To this end, we successively reduce the number of edges in the initial graph \( G = (V, E) \) (with \( |V| = n, |E| = m \)) without changing its nature. Finally, it is sufficient to see if the last graph (which will have no holes or dominoes, otherwise it could be simplified again) is a Meyniel graph or not. This will also be the answer \( G \).

The complexity of this algorithm will be of \( O(m^2 + mn) \). In order to simplify the notations, in this section the graphs will be identified with their set of vertices (sometimes represented as a list of disjoint subsets – this is the case of \( \Gamma_D \)).

Closely following the theoretical aspects presented in the previous section, we can describe the main steps of the algorithm as below (details on each step are given afterwards):

**Recognition Algorithm**

1. \( A = E \);
2. while \( A \neq \emptyset \) do
3. \( \mathrm{let } xy \in A \); \( A = A - xy \);
4. look for an even hole or domino \( C \) containing \( xy \);
   if not found, then return "\( G \) is not a Meyniel graph" or go to 10;
5. find \( T \);
6. \( \Gamma = C \); compute \( D_T(z) \) for every \( z \in C \); \( \Gamma_D = \{D_T(z) | z \in C \} \);
7. with a BFS-like algorithm:
   grow \( \Gamma \) and \( \Gamma_D \) as much as possible;
   verify that \( \Gamma \) is a skeleton; if not, return "\( G \) is not a Meyniel graph";
8. compute \( \Gamma', \Gamma'_1, \Gamma', \Gamma'_D \) or return "\( G \) is not a Meyniel graph";
9. compute \( S_{\Gamma'} \) (not containing \( xy \)) and \( G' \); \( G = G' \); \( A = A \cap E(G) \);
10. endwhile;
11. return "\( G \) is a Meyniel graph".

Obviously, the complexity is given by the loop "while" whose body is executed at most \( m \) times (since the cardinality of \( A \) reduces at every execution). The number of operations required by the steps 3–9 strongly depends on the data structures that are used; they are supposed to be suitable chosen to support easy access. For instance, \( G \) is represented both using the adjacency matrix and the adjacency lists
of neighbours (sorted in ascending order of vertices; that can be done in linear time using buckets-sort – see [1]). This allows us, on one hand, to know in constant time if a given pair of vertices is an edge or not, and on the other hand to quickly find all the neighbours of a given vertex. For the same reason, for any of the sets $C, T$ etc. a double representation may be used when needed.

With these general but important remarks, we describe the realisation of the steps contained in the loop, in order to have a complexity in $O(m + n)$ for each of them. The most frequently used algorithm is the breath-first-search, that we call BFS (or BFS$(z)$ when we want to specify the departure vertex $z$ of the traversal). It is well known that BFS needs $O(m_H + n_H)$ operations, for every graph $H$ with $m_H$ edges and $n_H$ vertices.

Steps 3 and 4: Take, initially, $xy$ the first edge in the list. Remove the common neighbours of $x, y$, all the edges joining a private neighbour of $x$ to a private neighbour of $y$, and the edge $xy$ itself. In the remaining graph, find (if it exists and using the BFS algorithm) a chordless path $P = \{x = z_1, z_2, \ldots, z_k = y\}$ and notice that the graph induced by $P$ in $G$ is a cycle of length at least 5, containing $xy$ and at most one chord. If it is odd, $G$ is not a Meyniel graph. If it is even, either it is a hole (and we are done), or a domino (and we are done once more), or contains the chordless cycle $\{z_2, z_3, \ldots, z_k, z_1\}$ of length at least 6. In this last case, we forget the initial edge $xy$ and recall $xy$ the edge $z_2 z_{k-1}$. The former $xy$ is no more removed from $A$, but the new one does. All these operations are easily done in $O(m + n)$ time.

Step 5: Obviously, $T \subseteq N_G(x)$. Then it is sufficient to verify, for every $w \in N_G(x)$, how many of its neighbours are in $C$, and then to compare this number with the cardinality of $C$.

Step 6: To compute $D_T(z)$ for every $z \in C$, we firstly find $G - T$. Then, for every $z \in C$, we perform the BFS algorithm in this graph starting with $z$ and considering only vertices which have not yet been placed in some $D_T(z')$. If the current vertex $u$ is adjacent to the (two or three) neighbours of $z$ on $C$, then $u$ is added to $D_T(z)$ and the BFS algorithm continues; otherwise, $u$ is ignored by BFS$(z)$ (as if it was not met). It is easy to see that any edge $uv$ in $G$ is used at most two times, and the maximum is realized when $u$ and $v$ are in $D_T(z), D_T(z')$ with $z \neq z'$; in this case, both the algorithms BFS$(z)$ and BFS$(z')$ use this edge. But the complexity remains in $O(m + n)$.

Remark 7 In this way, we put in $D_T(z)$ the vertices $u$ such that $N_G(u) \supseteq N_C(z)$, while by the definition of $D_T(z)$ we should have put the vertices for which the equality holds. In fact, by Claim 3 and recalling that we are working in $G - T$, this equality should hold. It will be verified later, after the final $\Gamma$ and all $D_T(z)$ are found, by verifying that all $D_T(z)$ are modules in $\Gamma_D$.

Step 7: To find $\Gamma$ and all the sets $D_T(z)$, we use the following algorithm in $G_1 = G - T - S$ (i.e. the connected component of $C$ in $G - T$). It corresponds to the construction we performed in the proof of Lemma 2. Recall that $\Gamma = C$ at the beginning of step 7.

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Remark 8 Some of the $K^i$ added to $\Gamma_D$ (and $x_i$ added to $\Gamma$) by the algorithm below may, in fact, correspond to the sets $C_T(z)$ which, in Lemma 2, are not connected to the skeleton. This is due to our construction which doesn’t test the connection to the skeleton, but only the adjacency to all $D_T(z)$. Therefore, we need to find here the 2-connected component of $\Gamma$ containing $C$, and to call it again $\Gamma$ in order to obtain the skeleton concerned by Lemma 2. This is done in $O(m + n)$ (see [14]). An update of $\Gamma_D$ is also needed.

The algorithm BFS requires $O(m_{Del} + n_{Del})$ operations for every $z$ (recall that the subgraphs are identified in this section with their set of vertices). Also, the connected components may be found (again using BFS) in $O(m_{Cr(z)} + n_{Cr(z)})$; since all the sets involved here are disjoint, the total complexity is in $O(m + n)$ for these operations. Removing $Del$ needs $O(m_{R_z} + n_{R_z})$ time, where $R_z = D_T(z) + Del$, for every $z$. Once more, we obtain a global complexity of $O(m + n)$ for this operation.

The test if $\Gamma$ is bipartite uses the BFS algorithm, whose complexity is linear.

Verifying if the sets $D_T(z)$ in $\Gamma_D$ are modules is again an easy task: it is sufficient to compare $N_{\Gamma_D}(z)$ to any $N_{\Gamma_D}(z')$, for any $z' \in D_T(z)$. This is easily done, for a fixed $z'$, in $O(|N(z')|)$ if the membership of a vertex to a set $D_T(u)$ is easy to test (suitable data structures exist) and if the adjacency lists of neighbours are sorted in ascending order of vertices.

Step 8: To compute $\Gamma'$, we consider every vertex $z \in \Gamma$ and, for every $z' \in D_T(z)$, calculate the number of its neighbours in $T$. It must be exactly $|T|$ for every $z'$ to deduce that $z \in \Gamma'$.

To find $\Gamma'$, the $O(m + n)$ algorithm in [10] may be applied. The algorithm in [14] is used again, once to verify whether $\Gamma'$ is 2-connected or not (and return “$G$ is not a Meyniel graph” if necessary), and once to find $\Gamma'$, the 2-connected component of $\Gamma'$ containing $C$. In the same time, the sets $D(z)$ are built by putting together the necessary $D_T(z)$.

aux = $\Gamma$;
while aux $\neq \emptyset$ do
  take $z \in$ aux; aux = aux $- z$;
  compute $W_T(z)$ and $C_T(z)$;
  using BFS, find the set $Del$ of vertices not in $D_T(z)$ accessible from $W_T(z)$;
  if $Del \cap \Gamma_D \neq \emptyset$ then return “$G$ is not a Meyniel graph”
  else remove $Del$ from $G_1$;
  compute the connected components $K^1, K^2, \ldots, K^p$ of $C(z)$;
  for $i = 1$ to $p$ do: arbitrarily take vertices $x_i \in K^i$; call $D_T(x_i) = K^i$;
  $\Gamma_D = \Gamma_D \cup \{D_T(x_i), i = 1, \ldots, p\}; \Gamma = \Gamma \cup \{x_i, i = 1, \ldots, p\}$;
  aux := aux $\cup \{x_i | i = 1, \ldots, p\}$;
endwhile;
if $\Gamma$ is not bipartite, then return “$G$ is not a Meyniel graph”;
for every $z \in \Gamma$ verify if $D_T(z)$ is a module in $\Gamma_D$; if not, return “$G$ is not a Meyniel graph”.

aux = $\Gamma$;
Step 9: The spanning tree $S_T$ is easily built using the BFS algorithm; the edge $xy$ will be ignored, since we want $S_T$ not to contain $xy$ (this is not really necessary, but it is more natural to destroy a cycle by removing the edge which “generated” it). The actualisation of $A$ is done by considering every edge $uv$ in $A$, finding (if they exist) $z_1, z_2$ such that $u \in D_T(z_1), v \in D_T(z_2)$, and verifying if $z_1 z_2$ is an edge of $T$ or not. For a fixed $uv$, this is done in constant time.

**Remark 9** It is easy to see that, in step 11 of the algorithm, $G$ must be a Meyniel graph: the remaining graph $G’$ has no chordless cycle on at least six vertices, and the induced $C_4$ are not contained in a house. To see this, let, by contradiction, $u_1, u_2, u_3, u_4$ be a $C_4$ included in the house $H$ also containing the vertex $u_5$ adjacent to $u_1, u_2$. Then, while considering $u_3 u_4 \in A$ at step 3, we should have obtained “$G$ is not a Meyniel graph” in step 4, a contradiction.

Thus, the steps 3–9 need $O(m + n)$ operations, therefore the whole algorithm for the Meyniel graphs recognition is done in $O(m^2 + mn)$.

7. Concluding remarks

Following [13] and [6], we define the parity graphs (resp. $i$-triangulated graphs) as the graphs in which every odd cycle has at least two crossing (resp. non-crossing) chords. It is easily seen that all these graphs are Meyniel graphs, therefore the general structure established in Section 2 is valid for these graphs, but it will have some particular aspects in any of the two cases:

- for parity graphs, the set $T$ (and therefore $S$) must be empty (otherwise any vertex in $t$ and any $P_t$ of $G’$ form an odd cycle with no crossing chords);
- for $i$-triangulated graphs, the sets $D(z)$ must have no edges (otherwise two adjacent vertices in this set and a chordless cycle of $G’$ containing $z$ form an odd cycle with no non-crossing chords); moreover, the graph induced by $T$ must be a clique (i.e. every pair of vertices forms an edge).

Statements similar to Theorem 1 are valid for both the parity and the $i$-triangulated graphs, yielding:

In the case of parity graphs, the proof of Theorem 1 is still valid, taking into account the fact that $T$ is an empty set (the two chords found at every step cross each other).

In the case of $i$-triangulated graphs, we can notice that for all $z$ in $T$, we have $D_T(z) = \{z\}$, so $W_T(z) = \emptyset$. Then $\Gamma_D = \Gamma$ and, because of Lemma 2 which guarantees that the sets $C_T(z)$ are not connected to $\Gamma_D$ in $G - D_T(z) - T$, we deduce that

\[(P) \quad \text{for every pair } x, y \text{ of vertices in } \Gamma, \text{ all the paths joining them in } G - T - S \text{ are contained in } \Gamma_D = \Gamma.\]

Now, the proof for the cases where some $x \in T$ exists on the cycle $C’$ is more simple, since $T$ is a clique. In the case where $C’$ is included in $G’ - T - S$ (respectively $G - T - S$), by the property $(P)$ and because at least two vertices of $C’$ are in $\Gamma$, we deduce that $V(C’) \subseteq V(\Gamma)$ and this contradicts the fact that $\Gamma$ is bipartite.
As a consequence, we can imagine to follow the same reasoning in order to obtain $O(m^2 + mn)$ recognition algorithms for parity and $i$-triangulated graphs; but we can notice that in step 11 of the algorithm we cannot decide directly that the graph is a parity graph, respectively an $i$-triangulated graph. Indeed, small bad structures may still be present: a cycle of length five with two non-crossing edges in the case of parity graphs, and a cycle of length five with two crossing edges in the case of $i$-triangulated graphs. Therefore, in step 12 of the algorithm we should verify that the remaining graph $G$ doesn't contain, respectively, these two structures. The first one is easy to find (if it exists) in $O(mn + n^2)$ by considering every vertex and verifying if the graph induced by its neighbours is $P_4$-free or not (see [4]). For the second one, it is sufficient to take any pair of non-adjacent vertices $x, y$ of $G$, and to test if the graph induced by their common neighbourhood contains a $P_3$ or not (i.e. if this graph is a multipartite graph; the modular decomposition algorithm may be applied); this time, $O(mn^2)$ operations are needed.

In this way, we can approach the recognition of these subclasses using the same tools as for the recognition of Meyniel graphs (see [2], [3] for different approaches).

References