PROPERLY ORDERABLE GRAPHS

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Abstract
In a graph $G = (V, E)$ provided with a linear order "<" on $V$, a chordless path with vertices $a, b, c, d$ and edges $ab, bc, cd$ is called an obstruction if both $a < b$ and $d < c$ hold. Chvátal [2] defined the class of perfectly orderable graphs (i.e., graphs possessing an acyclic orientation of the edges such that no obstruction is induced) and proved that they are perfect. We introduce here the class of properly orderable graphs which is a generalization of Chvátal’s class of perfectly orderable graphs: obstructions are forbidden only in the subgraphs induced by the vertices of an odd cycle. We prove the perfection of these graphs and give an $O(m^2 + mn + n)$ colouring algorithm.

1. Introduction.

Given a graph $G = (V, E)$, the chromatic number of $G$ is the minimum number of colours necessary to colour the vertices in $V$ in such a way that any two adjacent vertices have different colours. A clique is a set of pairwise adjacent vertices in $G$, while the clique number of $G$ designates the number of vertices in a largest clique. It is a simple observation that for any graph $G$ the clique number is less than or equal to the chromatic number. In the case when the equality holds for $G$ and each of its subgraphs, the graph is called perfect. It is called strongly perfect if every subgraph contains a stable set intersecting all its maximal cliques. Obviously, strongly perfect graphs are perfect.

Using the set of colours 1, 2, 3, ..., the most natural way to colour the vertices of a graph $G = (V, E)$ is to define a linear order "<" on $V$ and to process the vertices with respect to this order, giving them the smallest admissible colour. This way to assign colours to the vertices is called the greedy algorithm and the colouring obtained for $(G, <)$ is named a greedy-colouring of $G$. Certainly, a greedy-colouring is not always an optimal one, it may use a number of colours greater than the chromatic number of $G$. However, for every graph there exists an order of vertices yielding an optimal greedy-colouring; but this order cannot be found in polynomial time for every graph $G$. This is the reason why Chvátal [2] defined the class of perfectly orderable graphs, for which the greedy algorithm has the best behaviour. A graph is said to be perfectly
orderable if it admits a perfect order on $V$, that is, a linear order "$<$" such that no induced $P_4$ in $G$ with edges $ab, bc, cd$ has both $a < b$ and $d < c$ (such a $P_4$ is usually called an obstruction). Obviously, a perfect order on $V$ is also a perfect order on every $V' \subseteq V$, therefore the class of perfectly orderable graphs is hereditary. As shown by Chvátal, perfectly orderable graphs are strongly perfect and the greedy colouring determined by a perfect order is optimal.

We shall use this result of Chvátal to introduce the class of properly orderable graphs, which is larger than the class of perfectly orderable graphs, and to colour the members of this new class.

Our terminology is generally standard. The definitions not given here can be found in [1]. All the graphs we consider are finite and simple. We let $\omega(G)$ and $\chi(G)$ stand for the clique number and the chromatic number of the graph $G$. The set of colours we refer to is $1, 2, 3, \ldots$ and the notation we use to indicate the colours consists in capital letters. An $A$-vertex $v$ is a vertex coloured in $A$ ($v$ will be denoted $v[A]$). An $A$-vertex adjacent to a vertex $x$ is called an $A$-neighbour of $x$; $\text{ABcc}(x)$ is the connected component of $x$ (which is an $A$-vertex or a $B$-vertex) in the subgraph of $G$ induced by the vertices coloured in $A$ or $B$. Finally, an $\text{AB-interchange}$ is the colouring obtained by switching the colours $A$ and $B$ in $\text{ABcc}(x)$ for a vertex $x \in V$.

Using these new notations, a greedy-colouring of $(G, <)$ has the property that for every $R$-vertex $w$ in $G$, there exist vertices $w_1[1], w_2[2], \ldots, w_{R-1}[R-1]$, neighbours of $w$ in $G$, such that $w_i < w$ for all $i \in \{1, 2, \ldots, R-1\}$.

2. Main result.

Let us say that a graph $G = (V, E)$ is properly orderable if there exists a linear order on $V$ (also called a proper order) such that for every odd cycle $C$ of $G$, the subgraph induced in $G$ by $V(C)$ is perfectly ordered. A properly orderable graph provided with a proper order is said to be properly ordered.

Our aim is to prove that properly orderable graphs are a class of perfect graphs and to indicate a colouring algorithm. Before giving the main result, we introduce some more notations.

Consider $\tilde{G} = (V, \tilde{E})$ the directed graph obtained from $G$ by setting $ab \in \tilde{E}$ if and only if $ab \in E$ and $a < b$. For any vertex $a \in V$, $N^+(a)$ (resp. $N^-(a)$) is the set of vertices $b$ in $V$ such that $ab \in \tilde{E}$ (resp. $ba \in \tilde{E}$). The neighbourhood of the vertex $a$ is $N(a) = N^+(a) \cup N^-(a)$. If $N^+(a) = \emptyset$ (resp. $N^-(a) = \emptyset$) then $a$ is a sink (resp. a source). Obviously, the minimum and the maximum vertex of $G$ with respect to the linear order are a source and, respectively, a sink of $G$.

Meyniel [4] proved the perfectness of strict quasi-parity graphs, i.e., graphs whose subgraphs that are not cliques contain two nonadjacent vertices joined by no odd chordless path. Such a pair of vertices is called an even pair. On the other hand he showed that perfectly orderable graphs are strict quasi-parity graphs. The ideas in his proof are used below to establish the following result.

**Theorem 1.** Properly orderable graphs are strict quasi-parity graphs.

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Proof. Consider a connected graph $G$ (that is not a clique) and a proper order on its vertex set $V$. If for all $z$ in $G$, $N^-(z)$ is a clique, then $G$ is perfectly ordered and, by Meyniel's result, it has an even pair. Otherwise we use the induction on the number of vertices in $G$ to prove that:

$(P_1)$ If $w$ is the maximum vertex in $G$ such that $N^-(w)$ is not a clique, then there is an even pair of $G$ contained in $N^-(w)$.

For $|V| = 3, 4$ the property is true. We suppose it is valid for every properly orderable graph with less than $n$ vertices and we prove it for a graph $G$ with $n$ vertices. As before, let $x$ be the maximum vertex in $G$ and $N(x)$ its neighbourhood.

If $N(x)$ is a clique, then the maximum vertex $t$ in $G - x$ such that $N^-(t)$ is not a clique has the same property in $G$ and $(P_1)$ is valid. If $N(x)$ is not a clique, then let $y$ be a source of $N(x)$. Two cases could occur:

1. $N(x) - N(y) \neq \emptyset$. Let then $z$ be a source of $N(x) - N(y)$. We prove that $(y, z)$ is an even pair. Indeed, $y$ and $z$ are nonadjacent. Suppose that there is an odd chordless path joining $y$ and $z$ in $G$ and let $u$, resp. $v$ be the neighbours of $y$, resp. $z$ on this path. The graph induced by $\{x\} \cup V(P)$ is perfectly ordered, so $zv \in \vec{E}$ (otherwise the orientation of $vzxy$ imposes $vx \in E$ and $z$ is not a source in $N(x) - N(y)$).

Claim 1. If the odd chordless path $P = [y_1, y_2, y_3, \ldots, y_{2p-1}, y_{2p}]$ is perfectly ordered and $y_1y_2 \in \vec{E}$ then $y_{2p-1}y_{2p} \in \vec{E}$.

Proof. By induction on $i$ we can easily prove that $y_{2i-1}y_{2i} \in \vec{E}$ for all $i \in \{1, 2, \ldots, p\}$. Setting $i = p$ we obtain the conclusion of claim 1.

Then $uy \in \vec{E}$, since $zv \in \vec{E}$ and $P$ is perfectly ordered. Moreover, $ux \in \vec{E}$ (otherwise $u, y, x, z$ form an obstruction), therefore $y$ is not a source in $N(x)$, a contradiction.

2. $N(x) - N(y) = \emptyset$. Then $x$ is the maximum vertex in $G - y$ and $N_{G-y}(x) = N_{G-y}(x)$ is not a clique. By induction hypothesis, there is an even pair $(a, b)$ of $G - y$ entirely contained in $N_{G-y}(x)$. Therefore $(a, b)$ is also contained in $N(y)$ and there is no odd chordless path joining $a$ and $b$ in $G$.

By induction hypothesis, every subgraph $H$ of $G$ is strict quasi-parity and, since $G$ itself possesses an even pair, we obtain that $G$ is strict quasi-parity.

3. Colouring algorithm.

Consider a properly ordered graph $G$, the set of colours denoted by positive integers and the following colouring algorithm:
Algorithm COLOUR
for every $x$ in $V$ (in ascending order) do
  for every $z$ in $N^-(x)$ (in ascending order) do
    $A :=$ the smallest colour not present in $N^-(z) \cap N^-(x)$;
    $B :=$ the colour of $z$;
    if $A \neq B$ then AB-interchange in ABcc($z$)
  endfor;
  Colour($x$) := the smallest colour not present in $N^-(x)$
endfor.

Intuitively, the algorithm processes the vertices of $G$ following the proper order, uses a simulation of the greedy algorithm to colour the neighbourhood of the current vertex $x$ and gives to $x$ the smallest admissible colour. We shall prove:

Theorem 2. For a properly orderable graph, the colouring obtained by the algorithm COLOUR is optimal.

Proof. We use the induction on the number of vertices in $G = (V, E)$. If $|V| = 1, 2$ the theorem is valid. Suppose it is valid for every properly orderable graph with less than $n$ vertices and let prove it for any properly orderable graph $G$ with $n$ vertices.

Claim 2. Let $G = (V, E)$ be a properly orderable graph and "$<$" a proper order of $G$. For every $x$ in $V$, $N(x)$ is perfectly ordered.

Proof. Suppose the contrary and let $x \in V$ be a vertex such that $N(x)$ is not perfectly ordered. There exists a $P_4$ abcd in $N(x)$ with $ab, dc \in \bar{E}$. Then the cycle $C$ given by $x, a, b, c, d$ is an odd one and the subgraph induced in $G$ by its vertices is not perfectly ordered, a contradiction. \[\square\]

We denote by $x$ the maximum vertex in $G$. The neighbourhood $N(x)$ of $x$ is perfectly ordered and, by Chvátal’s result, the greedy algorithm based on this order gives an optimal colouring of $N(x)$. In general, this colouring of $N(x)$ is not necessarily extendible to an optimal colouring of $G$ itself. However, in our case, we shall prove that any optimal colouring of $G - x$ (by induction, there exists such a colouring) may be transformed into an optimal colouring of $G - x$ such that $N(x)$ is greedy-coloured with respect to "$<"$.

Consider an optimal colouring of $G - x$. By induction on $k$ (and with the notation $N_k(x)$ for the first $k$ vertices in $N(x)$) we show that

(P2) The colouring of $N_k(x)$ obtained by performing the interchanges in the algorithm COLOUR is identical to the greedy-colouring of $(N_k(x), <)$. 

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For $k=1$ the property holds. Suppose ($P_2$) is true for $k-1$ and let prove it for $k$. The next vertex in $N(x)$ corresponding to the order on $V$ is denoted by $z$ and its colour is $B$. Let $A$ be the smallest colour not present in the neighbourhood of $z$ in $N_{k-1}(x)$. Then $A \leq B$. In order to obtain a greedy-colouring in $N_k(x)$, $z$ must receive the colour $A$. If $A = B$, then $z$ is already coloured in $A$ and we have nothing to do. If $A < B$ two cases can occur: 

- $z$ has no $A$-neighbour in $G-x$. Then $z$'s colour may be changed in $A$.
- $z$ has at least one $A$-neighbour $w$ in $V \setminus \{x\} \setminus N_k(x)$. To obtain a greedy colouring in $N_k(x)$ it is necessary to perform the AB-interchange in $ABcc(z)$, provided that no vertex in $N_{k-1}(x)$ changes its colour during the interchange, that is, no vertex in $N(x)$ smaller than $z$ is contained in $ABcc(z)$. Assume the contrary, i.e. there is a vertex $q \in N_{k-1}(x) \cap ABcc(z)$.

**Claim 3.** There is at least one $A$-vertex $v \in N_{k-1}(x) \cap ABcc(z)$ adjacent to no $B$-vertex in $N_{k-1}(x)$ smaller than $v$.

**Proof.** Notice first that there is at least one $A$-vertex in $N_{k-1}(x) \cap ABcc(z)$. Indeed, if $q$ is an $A$-vertex, we are done. If $q$ is a $B$-vertex, there is an $A$-neighbour $r$ of $q$ in $N_{k-1}(x)$ such that $r < q$. Then $r$ is also in $ABcc(z)$ and we are done.

Now, let $v$ be the smallest $A$-vertex in $N_{k-1}(x) \cap ABcc(z)$. Then no $B$-neighbour of $v$ in $N(x)$ is smaller than $v$ (for otherwise we can find, as we have done before for $q[B]$, an $A$-neighbour of $x$ smaller than $v$ and contained in $ABcc(z)$). Thus claim 3 is proved. □

Since $z[B]$ and $v[A]$ (given by claim 3) are both contained in $ABcc(z)$, there is an odd chordless path $P$ in $ABcc(z)$ joining them. Let $t$ be the $B$-neighbour of $v$ on this path and $u$ the $A$-neighbour of $z$ on this path. The cycle given by $P$ and $x$ is an odd one, so the graph induced by $V(P) \cup \{x\}$ is perfectly ordered.

Let us notice that $tv \notin \vec{E}$. If this was the case, then $tx \in E$ (otherwise $t, v, x, z$ would induce an obstruction), therefore $t$ would be a $B$-neighbour of $v$ in $N_{k-1}(x)$ smaller than $v$. By claim 3, $v$ hasn’t got such neighbours, a contradiction. Thus $vt \in \vec{E}$. By claim 1, $uz \in \vec{E}$ and therefore $ux \in E$ (otherwise $v, x, z, u$ induce an obstruction). But then $u < z$ and $u$ is an $A$-neighbour of $z$ in $N_{k-1}(x)$. Since $A$ was supposed to be the smallest colour not present in the neighbourhood of $z$ in $N_{k-1}(x)$ we obtain a contradiction.

Conclusion: no vertex in $N(x)$ smaller than $z$ is contained in $ABcc(z)$. Therefore the AB-interchange in $ABcc(z)$ may be performed to obtain a greedy-colouring of $N_k(x)$.

For $k = |N(x)|$, we have a greedy-colouring of $N(x)$ with respect to the perfect order “$<$”, so we can apply Chvátal’s result to deduce that $N(x)$ is optimally coloured. The vertex $x$ may be then coloured with the smallest colour not present in $N(x)$ and the resulting colouring for $G$ uses a number of colours equal to the clique number. □

**Remark 1.** The class of properly orderable graphs strictly contains the class of perfectly orderable graphs. The graph $F$ in Fig.1 is an example of properly orderable graph that is not a perfectly orderable graph.

$F$ has the property that the two connected components of $F' = (V(F), E(F) \setminus \{bc\})$ are perfectly orderable and every perfect order imposes that at least one of the edges $ab, a'b$ (resp. $dc, d'c'$) is oriented toward $b$ (resp. toward $c$). Therefore $F$ is not perfectly orderable, but it is properly orderable since every obstruction is not contained in a subgraph of $F$ induced by the vertices of an odd cycle.
Remark 2. Properly orderable graphs are not necessarily strongly perfect graphs. To see this for $F$, it is sufficient to notice that an independent set $S$ which intersects all the maximal cliques would contain either $b$ or $c$ (say $b$) and then the $P_5$ containing $b$ would have a 2-clique with no vertex in $S$. Thus $F$ is not strongly perfect.

3. Complexity.

The colouring algorithm performs at most $m_x$ interchanges for every $x$, where $m_x$ is the number of edges joining $x$ to a vertex smaller than $x$. Every interchange involves at most $O(m)$ edges, so the total number of operations in interchanges is of $O(m^2)$. We must also add the number of operations involved by the other steps of the algorithm, that is $O(mn + n)$. The complexity of the algorithm is then $O(m^2 + mn + n)$.

Concerning the problem of recognizing properly orderable graphs, notice that its complexity is at least the complexity of recognizing perfectly orderable graphs. Indeed, given a recognizing algorithm for properly orderable graphs and a graph $G = (V, E)$, one can decide if $G$ is perfectly orderable or not as follows: add to $G$ a vertex joined to all the vertices in $V$; verify if the new graph belongs to the class of properly orderable graphs; if the answer is "yes", then $G$ is perfectly orderable; otherwise, $G$ is not perfectly orderable.

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References.