

k -Neighborhood Broadcasting*

GUILLAUME FERTIN

IRIN, Université de Nantes, France

ANDRÉ RASPAUD

LaBRI, Université Bordeaux 1, France

Abstract

Broadcasting refers to the task whereby a node in a network, knowing a piece of information, must transmit it to all the other nodes. In this paper we consider a generalized form of broadcasting, that we call k -neighborhood broadcasting. It consists in the following : a node u in the network has to send its information to all the nodes which are at distance less than or equal to k from u .

We study k -neighborhood broadcasting (or k -NB for short) in paths, trees, cycles, 2-dimensional grids and 2-dimensional tori under the store and forward, 1-port, unit cost model. For most of these families, we give the optimal k -NB time ; if not, the optimal k -NB time is given within an additive constant never exceeding 2.

Keywords

Broadcasting, communications, networks.

1 Introduction and Preliminaries

We model a network by a graph, where the vertices and the edges represent respectively the processors and the communication links of the network. In these networks, broadcasting consists, for a source node u of the network holding a piece of information, in sending this information to all the other nodes in the network. Broadcasting has been widely studied in the previous years, and has given rise to several books and articles. We refer to [10, 5, 3, 11] for comprehensive surveys on this problem.

In this paper, we consider a generalized version of this problem, that we call k -neighborhood broadcasting (or k -NB for short) ; here, broadcasting is realized at distance k in the graph G , $1 \leq k \leq D(G)$, where $D(G)$ denotes the diameter of G . More formally, k -NB from u consists in sending u 's information

*Part of this research was done while the first author was visiting Simon Fraser University, Vancouver, Canada.

all the nodes which are within distance k from u .

The model we use here is *store-and-forward* (messages go to their destination by transiting in the intermediate nodes ; that is, communications always take place between neighbors), *1-port* (at each given time, a node can communicate with at most one of its neighbors) and *unit cost* (a communication between two nodes takes one time unit, or *round*).

When $k = D(G)$, then the k -NB problem is the usual broadcasting problem, and as mentioned earlier, a wide literature on the subject exists. We note that this problem has also been studied in the particular case when $k = 1$ in two different families of graphs, namely hypercubes [2, 1] and star graphs [9].

We note that even in the case $k = 1$, the time to achieve 1-NB in a graph is not necessarily simply equal to the maximum degree $\Delta(G)$ of the graph. For example, for this, consider the wheel graph $W_{1,n}$ that consists of a cycle of order n , C_n , and we add a “center” vertex c , connected to all the vertices of C_n . In that case, it can be seen that the time to achieve 1-NB from any vertex different from c is equal to 2, and the time to achieve 1-NB from c is equal to $\left\lceil \frac{1+\sqrt{1+4(n-1)}}{2} \right\rceil$, the degree of c , $d(c)$, satisfies $d(c) = \Delta(W_{1,n}) = n$.

Let us also mention that another usual type of communication in networks is gossiping (where every node must send its information to every other node). This can be similarly generalized to k -neighborhood gossiping (or k -NG). In the particular case $k = 1$, the 1-NG problem has been studied in hypercubes [6] and star graphs [9] ; for general k , a study of k -NG, similar to the one presented in this paper, has been undertaken in [7].

In this paper, we give bounds on the optimal time for k -NB in the following families of graphs : paths, trees, cycles, 2-dimensional grids and tori. In those families, we determine the optimal k -NB time ; if not, we give bounds on the optimal k -NB time, which differ by an additive constant not exceeding 1. In Section 2, we give some necessary definitions and determine some general properties about the minimum time to achieve k -NB in a graph G . In Sections 3 and 4, we study k -NB in paths, trees of maximum degree Δ and diameter D , cycles, 2-dimensional grids and 2-dimensional tori.

2 Generalities

We denote by $t_{NB}(G, u, k)$ the minimum time to achieve k -NB from vertex u in the graph G under our model. We call the k -neighborhood broadcast time of G , or $t_{NB}(G, k)$, the maximum $t_{NB}(G, u, k)$ over all $u \in V(G)$. Let also $N_k(u)$ be the number of vertices which are at distance less than or equal to k from u (included), and $N_k(G) = \max_{u \in V(G)} N_k(G, u)$. We then have the following

Property 1 For any graph G and for any $1 \leq k \leq D(G)$,

$$t_{NB}(G, k) \geq \max\{k, \lceil \log_2(N_k(G)) \rceil\}$$

Proof. Any vertex u in a graph G which has to inform t vertices (u in needs at least $\lceil \log_2(t) \rceil$ time units in our model (at each time unit, the number of informed nodes can at most double). Since in k -NB, there exists at least one vertex u in G that must inform $N_k(G)$ vertices, we then get $t_{NB}(G, k) \geq \lceil \log_2(N_k(G)) \rceil$. Finally, it is easy to see that $t_{NB}(G, k) \geq k$: indeed, $k \leq D(G)$, hence there exists two vertices u and v at distance k . Since u needs to broadcast its information to v , we have $t_{NB}(G, u, k) \geq k$, and thus $t_{NB}(G, k) \geq k$.

Property 2 Let G be a bipartite graph and u a vertex of G . For any $1 \leq k \leq D(G)$ and any $p \geq 0$, we denote by $N(k, p)$ the maximum number of vertices at distance exactly k from u that can be informed by u within $k + p$ rounds. In this case,

$$N(k, p) \leq \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} \binom{k+p}{k+2i}.$$

Proof. Let $u \in V(G)$. If u is to inform a vertex v at distance exactly k from u within $k + p$ rounds, it can either use a shortest path, or it can reach v via a longer path. However, since G is a bipartite graph, the latter only happens when the length of this path is of the same parity as k . Thus, in each case, a path of length $k + 2i$, with $0 \leq i \leq \lfloor \frac{p}{2} \rfloor$, can be used to reach a vertex at distance k . In order to determine how many such paths are possible, it suffices to note that the number of bipartite sequences from u to v along a path of length $k + 2i$, using rounds between k and $k + p$, is equal to the binomial $\binom{k+p}{k+2i}$. Summing this term over all possible values of i (i.e., $0 \leq i \leq \lfloor \frac{p}{2} \rfloor$), we get the result.

Thanks to the above property, and thanks to a well-known observation (see [1]), we can deduce the following proposition, that we will use several times in the rest of the paper.

Proposition 1 For any graph G , let u be a vertex of G . For any $1 \leq k \leq D(G)$, let $s_k(u)$ be the number of vertices at distance exactly k from u . Then :

- (a) If $s_k(u) \geq 2$, $t_{NB}(G, k) \geq k + 1$;
- (b) If G is bipartite and $s_k(u) \geq k + 2$, $t_{NB}(G, k) \geq k + 2$.

In the following, we denote by $G \square H$ the cartesian product of graph G and H .

Proposition 2 For any graphs G and H , and for any $1 \leq k \leq \min\{D(G), D(H)\}$, $t_{NB}(G \square H, k) \leq t_{NB}(G, k) + t_{NB}(H, k)$.

Proof. The broadcast scheme we propose here from any vertex $(u, v) \in V$ is the following : achieve a k -NB from (u, v) to all the vertices of the forest that are in the copy of G containing (u, v) . Then, achieve a k -NB from the vertices (u', v) reached by the first part of the scheme, to every vertex within distance k from (u', v) , that lies in the copy of H containing (u', v) . This scheme respects the 1-port model, and is achieved in time $t_{NB}(G, k) + t_{NB}(H, k)$. Moreover, this is a k -NB scheme : indeed, it can be easily seen that every vertex within distance k from (u, v) is informed.

3 Paths, Trees and Cycles

3.1 Paths

Theorem 1 *Let P_n be the path of order n . For any n and $1 \leq k \leq n - 1$,*

$$t_{NB}(P_n, k) = \begin{cases} k & \text{if } k > \lfloor \frac{n-1}{2} \rfloor \\ k + 1 & \text{otherwise} \end{cases}$$

Proof. By Property 1, we know that for any $1 \leq k \leq D(P_n)$, that is $1 \leq k \leq n - 1$, we have $t_{NB}(P_n, k) \geq k$. Moreover, let us show that when $k \leq \lfloor \frac{n-1}{2} \rfloor$, there exists at least one vertex u that has to inform 2 vertices at distance k , which implies by Proposition 1(a) that $t_{NB}(P_n, u, k) \geq k + 1$. For this, denote by $u_1, u_2 \dots u_n$ the vertices of P_n , from left to right, and let us consider the vertex $u_{\lfloor \frac{n-1}{2} \rfloor + 1}$. It has $\lfloor \frac{n-1}{2} \rfloor$ vertices on “its left” (that is, vertices denoted by u_i for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$), and $n - 1 - \lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n-1}{2} \rceil \geq \lfloor \frac{n-1}{2} \rfloor$ vertices on “its right” (that is, vertices denoted by u_j for $\lfloor \frac{n-1}{2} \rfloor + 1 \leq j \leq n$). Note also that for any $\lfloor \frac{n-1}{2} \rfloor < k \leq n - 1$, then for any $u_i \in V(P_n)$, there is at most one vertex at distance k from u_i .

Now, for any vertex u_i in P_n and every $1 \leq k \leq n - 1$, the broadcast scheme is as follows : if u_i has two vertices at distance k , then broadcast along (resp. left) edges from round 1 to round k (resp. from round 2 to round $k + 1$). If u_j is the only vertex at distance k from u_i , then if $j > i$ (resp. $j < i$) broadcast from round 1 to round k on the right (resp. left) edges, and from round 2 to round $|j - i| + 1$ on the left (resp. right) ones. If there is no vertex at distance k from u_i then broadcast to the left from round 1, and to the right from round 2. In all the cases, it can be seen that the broadcast scheme we propose here meets the appropriate lower bound.

Non surprisingly, in the case $k = D(P_n) = n - 1$, we obtain the same result as in [4] concerning the usual broadcast problem, that is $t_{NB}(P_n, n - 1) = n$.

3.2 Trees of maximum degree Δ and diameter D

In this section, we determine t_{NB} for the family $\mathcal{T}_{\Delta,D}$ of trees with maximum degree Δ and diameter D . We denote by $t_{NB}(\mathcal{T}_{\Delta,D}, k)$ the maximum k -neighborhood broadcast time over all the trees $T \in \mathcal{T}_{\Delta,D}$.

Definition 1 ($\mathbb{T}_0(\Delta, D)$ and $\mathbb{T}_1(\Delta, D)$) *For any fixed Δ and even D , we define $\mathbb{T}_0(\Delta, D)$ the unique tree rooted at r such that the leaves lie at distance $D/2$ from r , and such that all the vertices (except the leaves) are of degree Δ .*

For any odd $D = 2D' + 1$, $\mathbb{T}_1(\Delta, D)$ can be obtained from 2 copies of $\mathbb{T}_0(\Delta, 2D')$, where we delete a branch incident to the root r_i in each copy ($i \in \{1, 2\}$), and where we then connect the two roots r_1 and r_2 by an edge.

Examples of such trees are given in Figure 1, where we show respectively $\mathbb{T}_0(4, 4)$ and $\mathbb{T}_1(4, 5)$.

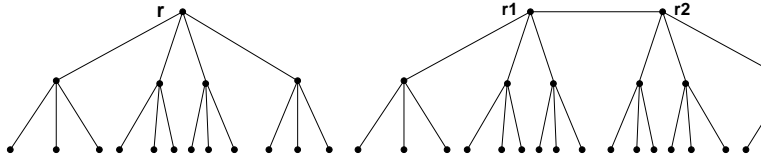


Figure 1: $\mathbb{T}_0(4, 4)$ (left) and $\mathbb{T}_1(4, 5)$ (right).

Clearly, $\mathbb{T}_0(\Delta, D) \in \mathcal{T}_{\Delta,D}$ (resp. $\mathbb{T}_1(\Delta, D) \in \mathcal{T}_{\Delta,D}$) for any fixed Δ and even D (resp. odd D). More generally, for any fixed Δ , it is easy to see that for any D , any tree $T \in \mathcal{T}_{\Delta,D}$ is a subtree of \mathbb{T}_0 , and for any odd D , any tree $T \in \mathcal{T}_{\Delta,D}$ is a subtree of \mathbb{T}_1 .

We also note that for odd D , $\mathbb{T}_1(\Delta, D)$ can be seen as a copy of $\mathbb{T}_0(\Delta, D)$ in which half of the leaves (in either the left or right subtree of r) has been removed.

Lemma 1 *For any fixed Δ and D , and for any $1 \leq k \leq \lfloor \frac{D}{2} \rfloor$:*

- (a) *if D is even, $t_{NB}(\mathbb{T}_0(\Delta, D), k) \geq k \cdot (\Delta - 1) + 1$;*
- (b) *if D is odd, $t_{NB}(\mathbb{T}_1(\Delta, D), k) \geq k \cdot (\Delta - 1) + 1$.*

Proof. Suppose first D is even, and let us show that for any $1 \leq k \leq \lfloor \frac{D}{2} \rfloor$, $t_{NB}(\mathbb{T}_0(\Delta, D), r, k) \geq k \cdot (\Delta - 1) + 1$, where r is the root of $\mathbb{T}_0(\Delta, D)$. This is done by induction on k . When $k = 1$, it takes time Δ for r to inform all vertices at distance k . Now suppose that $t_{NB}(\mathbb{T}_0(\Delta, D), r, k) \geq k \cdot (\Delta - 1) + 1$ for a fixed k , and let us show that this implies $t_{NB}(\mathbb{T}_0(\Delta, D), r, k+1) \geq (k+1) \cdot (\Delta - 1) + 1$, provided that $k+1 \leq \frac{D}{2}$. There exists a vertex v in $\mathbb{T}_0(\Delta, D)$ at distance $k+1$ from r which is informed after round $k \cdot (\Delta - 1) + 1$. Moreover, since $k+1 \leq \frac{D}{2}$, v has $\Delta - 1$ sons in $\mathbb{T}_0(\Delta, D)$ which are at distance $k+1$ from r . Hence it will

least $\Delta - 1$ more rounds for v to inform its $\Delta - 1$ sons in $\mathbb{T}_0(\Delta, D)$, and have $t_{NB}(\mathbb{T}_0(\Delta, D), r, k + 1) \geq k \cdot (\Delta - 1) + 1 + \Delta - 1$. Thus the result is proved by induction on k .

Now suppose D is odd. In that case, for any $k \leq \lfloor \frac{D-1}{2} \rfloor$, it is easy to see that the subtree of $\mathbb{T}_1(\Delta, D)$ induced by the vertices at distance less than or equal to k from r_1 is isomorphic to $\mathbb{T}_0(\Delta, 2k)$, with $r = r_1$. Thus, since by Lemma 1(a), we have that $t_{NB}(\mathbb{T}_0(\Delta, 2k), r, k) \geq k \cdot (\Delta - 1) + 1$, we get $t_{NB}(\mathbb{T}_1(\Delta, D), r, k) \geq k \cdot (\Delta - 1) + 1$. Hence $t_{NB}(\mathbb{T}_1(\Delta, D), k) \geq k \cdot (\Delta - 1) + 1$, and the result is proved.

Lemma 2 For any fixed Δ, D and $1 \leq k \leq \lfloor \frac{D}{2} \rfloor$:

- (a) if D is even, $t_{NB}(\mathbb{T}_0(\Delta, D), k) \leq k \cdot (\Delta - 1) + 1$;
- (b) if D is odd, $t_{NB}(\mathbb{T}_1(\Delta, D), k) \leq k \cdot (\Delta - 1) + 1$.

Proof. First suppose D is even. Let v be any vertex of $\mathbb{T}_0(\Delta, D)$. The main idea here is to note that the subtree of $\mathbb{T}_0(\Delta, D)$ induced by the vertices at distance less than or equal to k from v is itself a subtree T of $T_v = \mathbb{T}_0(\Delta, 2k)$. T_v is rooted at v (cf. Figure 2, where $\Delta = 4, D = 6$ and $k = 3$). However, in T_v , we know that we can achieve k -NB from v within $k \cdot (\Delta - 1) + 1$ rounds. Hence, for any vertex $v \in \mathbb{T}_0(\Delta, D)$, we have $t_{NB}(\mathbb{T}_0(\Delta, D), v, k) \leq k \cdot (\Delta - 1) + 1$. This proves the result when D is even.

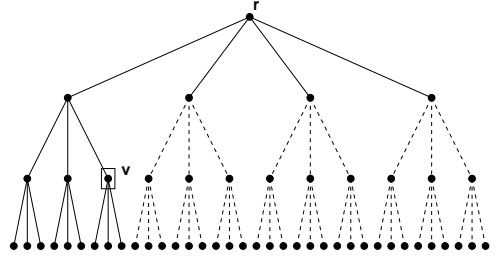


Figure 2: In $\mathbb{T}_0(4, 6)$, we construct T rooted at v , with $k = 3$.

Similarly, when D is odd, for any $v \in \mathbb{T}_1(\Delta, D)$, the subtree T' induced by the set of vertices that lie within distance k of v in $\mathbb{T}_1(\Delta, D)$ is itself a subtree $T'_v = \mathbb{T}_0(\Delta, 2k)$, where T'_v is rooted at v . Since we know that in T'_v (and thus in $\mathbb{T}_1(\Delta, D)$) it is possible to achieve k -NB from v within $k \cdot (\Delta - 1) + 1$ rounds, we therefore have $t_{NB}(\mathbb{T}_1(\Delta, D), v, k) \leq k \cdot (\Delta - 1) + 1$ for any v in $\mathbb{T}_1(\Delta, D)$, and thus the result is also proved for odd D .

Lemma 3 For any fixed Δ, D and $\lfloor \frac{D}{2} \rfloor + 1 \leq k \leq D$:

- (a) if D is even, $t_{NB}(\mathbb{T}_0(\Delta, D), k) \geq \frac{D}{2} \cdot (\Delta - 2) + k$;

(b) if D is odd, $t_{NB}(\mathbb{T}_1(\Delta, D), k) \geq \lfloor \frac{D}{2} \rfloor \cdot (\Delta - 2) + k$.

Proof. Suppose first D is even, and take any vertex v that lies at distance k from the root r in $\mathbb{T}_0(\Delta, D)$ (such a vertex exists, since $k \leq D$). Actually, we can choose v such that it must broadcast to all the other vertices in the tree. Let r_v be the son of r that is on the path from v to r , and let T' be the subtree of $\mathbb{T}_0(\Delta, D)$ containing r_v , and obtained by deleting the edge rr_v . Any vertex not in T' must be informed via r ; that is, r must inform all the vertices of $T_r = \mathbb{T}_0(\Delta, D) - T'$. Moreover, T_r cannot be informed before round $R_1 = k - \frac{D}{2}$. Now, in order for r to inform all the vertices of T_r , it can be seen that we need at least $R_2 = \frac{D}{2}(\Delta - 1)$ rounds. So, from R_1 and R_2 , we get $t_{NB}(\mathbb{T}_0(\Delta, D), v, k) \geq \frac{D}{2} \cdot (\Delta - 2) + k$.

Now let us assume D is odd. In that case, we use the same kind of argument. Let T_1 (resp. T_2) be the subtree of $\mathbb{T}_1(\Delta, D)$ that contains r_1 (resp. r_2) obtained by removing the edge r_1r_2 , and let v be a vertex lying at distance k from r_1 in T_1 . v has been chosen so that it must broadcast to all the vertices in T_1 . In particular all the vertices of T_2 must be informed via r_1 and r_2 . Moreover, let us denote by r'_1 the son of r_1 that lies on the path from v to r_1 , and if we denote by T' the subtree containing r'_1 obtained by deleting the edge r'_1r_1 , all the vertices of $T_1 - T'$ must also be informed via r_1 . Whatever the strategy used, and in particular whatever round is used on edge r_1r_2 , it can be shown that there exists a vertex u in $T_2 - T'$ that cannot be informed by u in less than $k - \lfloor \frac{D}{2} \rfloor + (\Delta - 1)\lfloor \frac{D}{2} \rfloor$ rounds. In other words, we have $t_{NB}(\mathbb{T}_1(\Delta, D), v, k) \geq \lfloor \frac{D}{2} \rfloor \cdot (\Delta - 2) + k$ and thus the result is proved.

Lemma 4 For any fixed Δ, D and $\lfloor \frac{D}{2} \rfloor + 1 \leq k \leq D$:

(a) if D is even, $t_{NB}(\mathbb{T}_0(\Delta, D), k) \leq \frac{D}{2} \cdot (\Delta - 2) + k$;

(b) if D is odd, $t_{NB}(\mathbb{T}_1(\Delta, D), k) \leq \lfloor \frac{D}{2} \rfloor \cdot (\Delta - 2) + k$.

Proof. Suppose first D is even, and let $\frac{D}{2} + 1 \leq k \leq D$. First, we know that any vertex in $\mathbb{T}_0(\Delta, D)$ is at distance at most $\frac{D}{2}$ from r . Hence, any k -NB from r with $k \geq \frac{D}{2} + 1$ is equivalent to a $\frac{D}{2}$ -NB from r , and thus takes at most $\frac{D}{2}(\Delta - 1) + k$ rounds. Since $k \geq \frac{D}{2} + 1$, we see that we always have $\frac{D}{2}(\Delta - 1) + 1 \leq k + \frac{D}{2}$. In other words, any k -NB from r always satisfies the inequality (a) of Lemma 4 above.

Now, for any vertex $v \neq r$ in $\mathbb{T}_0(\Delta, D)$, let $dist(v, r)$ be its distance from the root r , and let the *local root* lr be defined as follows : $lr = r$ if $dist(v, r) \leq k$ and lr is the vertex lying at distance $k - \frac{D}{2}$ from v on the path from v to r otherwise. Two cases then arise :

(a) $lr = r$. In that case, this means that v must broadcast to all the

in the tree. Let v inform r using rounds $1, 2, \dots, \text{dist}(v, r)$. Let us denote by r' the son of r that lies on the path from v to r , and by T' the tree rooted at r' that is obtained by removing the edge rr' . Now, r informs all the vertices of $\mathbb{T}_0(\Delta, D) - T'$ starting from round $\text{dist}(v, r) + 1$. This is achieved using $\frac{D}{2}(\Delta - 2) + k$ rounds on the whole. During that time, r' broadcasts in T' ($\frac{D}{2} - 1$), with the constraint that the path from v to r is already allowed. Hence, globally, the broadcast scheme uses at most $\frac{D}{2}(\Delta - 2) + k$ rounds.

(b) $lr \neq r$. In that case, v does not need to broadcast to all the vertices of the tree : its k -neighborhood does not contain every vertex in the tree. Hence, using the same notation as previously, it must broadcast to all the vertices of $\mathbb{T}_0(\Delta, D) - T'$ and to all the vertices lying at distance less than or equal to $\frac{D}{2}$ from lr in $\mathbb{T}_0(\Delta, D) - T'$. One can also see that the depth of T' is less than or equal to $\frac{D}{2} - 1$, hence, globally the broadcast scheme described in the previous case remains the same, except that lr is substituted to r . The maximum number of rounds used here is also $\frac{D}{2}(\Delta - 2) + k$, and thus the result is proved.

When D is odd, we use a very similar kind of argument to show that k -NB can be achieved from any vertex v in $\lfloor \frac{D}{2} \rfloor \cdot (\Delta - 2) + k$. \square

Theorem 2 $t_{NB}(\mathcal{T}_{\Delta, D}, k) = \begin{cases} k \cdot (\Delta - 1) + 1 & \text{if } 1 \leq k \leq \lfloor \frac{D}{2} \rfloor \\ \lfloor \frac{D}{2} \rfloor \cdot (\Delta - 2) + k & \text{if } \lfloor \frac{D}{2} \rfloor < k \leq D \end{cases}$

Proof. We know that any tree $T \in \mathcal{T}_{\Delta, D}$ is a subtree of $\mathbb{T}_0(\Delta, D)$ when D is even (resp. a subtree of $\mathbb{T}_1(\Delta, D)$ when D is odd). Thus we have $t_{NB}(T, k) \leq t_{NB}(\mathbb{T}_0(\Delta, D), k)$ for any $T \in \mathcal{T}_{\Delta, D}$ with even D , and $t_{NB}(T, k) \leq t_{NB}(\mathbb{T}_1(\Delta, D), k)$ for any $T \in \mathcal{T}_{\Delta, D}$ with odd D .

Moreover, by Lemmas 1 to 4 we have :

- If D is even, then $t_{NB}(\mathbb{T}_0(\Delta, D), k) = \begin{cases} k \cdot (\Delta - 1) + 1 & \text{when } 1 \leq k \leq \frac{D}{2} \\ \frac{D}{2} \cdot (\Delta - 2) + k & \text{when } \frac{D}{2} < k \leq D \end{cases}$

- If D is odd, then

$$t_{NB}(\mathbb{T}_1(\Delta, D), k) = \begin{cases} k \cdot (\Delta - 1) + 1 & \text{when } 1 \leq k \leq \frac{D-1}{2} \\ \frac{D-1}{2} \cdot (\Delta - 2) + k & \text{when } \frac{D-1}{2} < k \leq D \end{cases}$$

Hence the result.

Remark: We note that Theorem 1 is a particular case of Theorem 2. Indeed, if we suppose $\Delta = 2$, $\mathcal{T}_{\Delta, D}$ only contains one tree, more precisely P_{D+1} . Putting now $n = D + 1$ gives directly Theorem 1.

We also note that there exists several results concerning the (usual) broadcast time in some specific types of trees, such as complete Δ -ary trees of depth h (for instance [3]). However, these results cannot be compared to ours, since in complete Δ -ary trees, every vertex not being a leaf has Δ sons. This means that in complete Δ -ary trees, the root is of degree Δ , and every other vertex not a leaf is of degree $\Delta + 1$; in our case, every vertex not a leaf (root included) is of degree Δ . This is not to mention that complete Δ -ary trees of depth h are of diameter $D = 2h$, and that in our case we do not restrict D to be even.

3.3 Cycles

Theorem 3 *Let C_n be the cycle of order n . For any $n \geq 3$ and any $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$*

$$t_{NB}(C_n, k) = \begin{cases} k & \text{if } n \text{ is even and } k = \frac{n}{2} \\ k + 1 & \text{otherwise} \end{cases}$$

Proof. For any n , C_n is vertex-transitive, thus we can focus on broadcasting from a given vertex $v \in V(C_n)$. Suppose first n is odd. In that case, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, there are two vertices at distance k from v , and thus by Proposition 1(a) $t_{NB}(C_n, k) \geq k + 1$. When n is even, there are also 2 vertices at distance k from v , except in the case $k = \frac{n}{2}$, where there is only one such vertex. Thus, if n is even, $t_{NB}(C_n, k) \geq k + 1$ when $k \neq \frac{n}{2}$ by Proposition 1(a) and $t_{NB}(C_n, \frac{n}{2}) \geq \frac{n}{2} + 1$ otherwise.

Now let us show that these bounds are tight. For this, let v broadcast information at distance k in C_n as follows : at round 1, v sends its information to its neighbor, and in the following rounds, every newly informed vertex sends the information to its still uninformed neighbor. Following this scheme, we can see that any k -NB in C_n is achieved in $k + 1$ rounds, except in the case $k = \frac{n}{2}$ and $k = \frac{n}{2}$, where k rounds suffice (since, as mentioned above, for any v there exists only one vertex v' at distance k from v).

Here again, we note that in the case $k = D(C_n) = \lfloor \frac{n}{2} \rfloor$, we obtain the known result concerning (usual) broadcasting in cycles ; that is, $t_{NB}(C_n, \lfloor \frac{n}{2} \rfloor) = \lfloor \frac{n}{2} \rfloor + 1$.

4 2-Dimensional Grids and Tori

In the following, $G(n_1, n_2)$ (resp. $TG(n_1, n_2)$) will denote the 2-dimensional grid (resp. torus) with n_1 rows and n_2 columns. Due to symmetries, we always suppose $n_1 \geq n_2$. We also denote here by $D(G(n_1, n_2))$ the diameter of the 2-dimensional grid $G(n_1, n_2)$, and by $D(TG(n_1, n_2))$ the diameter of the 2-dimensional torus $TG(n_1, n_2)$. Thus we have $D(G(n_1, n_2)) = n_1 + n_2$

$$D(TG(n_1, n_2)) = \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor.$$

4.1 2-Dimensional Grids

Proposition 3 (1-NB in 2-dim. Grids) For any integers n_1, n_2 with $n_1 \geq 2$ and $n_2 \geq 2$,

$$t_{NB}(G(n_1, n_2), 1) = \begin{cases} 2 & \text{if } n_1 = n_2 = 2 \\ 3 & \text{otherwise} \end{cases}$$

Proof. When $n_1 = n_2 = 2$, it is easy to see that 2 rounds are necessary and sufficient for any vertex to broadcast at distance 1. In all the other cases, there exists in the grid at least one vertex v for which the number of vertices at distance 1 from v is greater than or equal to 3. Since $G(n_1, n_2)$ is a bipartite graph, by Proposition 1(b) we conclude that $t_{NB}(G(n_1, n_2), 1) \geq 3$. However, it is easy to see that 3 rounds are sufficient in that case (cf. for this Figure 3(left)).

Proposition 4 (2-NB in 2-dim. Grids) For any integers n_1, n_2 with $n_1 \geq 3$ and $n_2 \geq 3$,

$$t_{NB}(G(n_1, n_2), 2) = \begin{cases} 2 & \text{if } n_1 = n_2 = 2 \\ 3 & \text{if } n_1 \in \{3, 4\} \text{ and } n_2 = 2 \\ 4 & \text{if } n_1 \geq 5 \text{ and } n_2 = 2 \text{ or if } n_2 \in \{3, 4\} \\ 5 & \text{otherwise (that is, if } n_1 \geq n_2 \geq 5) \end{cases}$$

Sketch of Proof : The proof relies here on detailing all the possible cases. For that, for fixed n_2 , for any $n_1 \geq 5$, the result is the same as for $n_1 = 5$. For any $n_2 \leq 4$, by a rather tedious case by case analysis, the above result can be deduced. If $n_1 \geq n_2 \geq 5$, we note that there exists a vertex v in $G(n_1, n_2)$ that must inform $4k = 8$ vertices at distance $k = 2$. By Property 2, if we are able to inform $k + 2 = 4$ rounds, then v can inform at most $\frac{(k+2)(k+1)}{2} + 1 = 7$ vertices at distance 2. Thus, in that case $t_{NB}(G(n_1, n_2), 2) \geq 5$; it is easy to see that 5 rounds are necessary for any vertex in $G(n_1, n_2)$ (with $n_1 \geq n_2 \geq 5$) to broadcast its information at distance 2 (cf. for instance Figure 3(middle)).

Proposition 5 (3-NB in 2-dim. Grids) For any integers $n_1 \geq 3$ and $n_2 \geq 3$, with $n_1 \geq n_2$,

$$t_{NB}(G(n_1, n_2), 3) = \begin{cases} 3 & \text{if } n_1 = 3 \text{ and } n_2 = 2 \\ 4 & \text{if } n_1 \in \{4, 5, 6\} \text{ and } n_2 = 2 \\ & \text{or if } n_1 \in \{3, 4\} \text{ and } n_2 = 3 \text{ or if } n_1 = n_2 = 4 \\ 5 & \text{if } n_1 \geq 7 \text{ and } n_2 = 2 \text{ or if } n_1 \geq 5 \text{ and } n_2 \in \{3, 4\} \\ & \text{or if } n_1 \in \{5, 6\} \text{ and } n_2 = 5 \\ 6 & \text{otherwise} \end{cases}$$

Sketch of Proof : Here again, we detail all the possible cases up to $G(7, 7)$. However, when $n_1 \geq n_2 \geq 7$, by Property 2 we can show that $k + 3 = 6$ rounds are necessary to achieve a k -NB from any of the vertices of $G(n_1, n_2)$; as in the

case, it is easy to show that 6 rounds are enough (cf. for instance Figure 3). \square

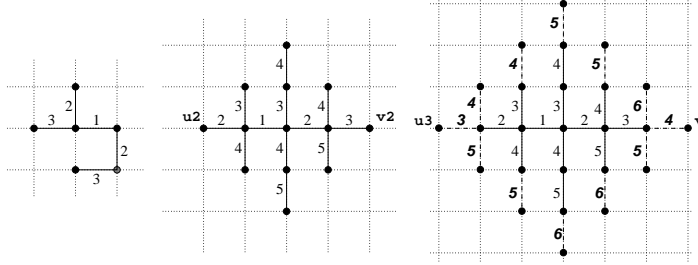


Figure 3: 1-NB (left), 2-NB (middle) and 3-NB (right) in $G(n_1, n_2)$

Proposition 6 (*k*-NB in 2-dim. Grids) *For any integers n_1, n_2 with $n_1 \geq 4$ and $n_1 + n_2 \geq 6$, let $\text{eq}(n_1, n_2) = 1$ if $n_1 = n_2$ and 0 otherwise. Let $a = \lfloor \frac{2n_2 + n_1}{3} \rfloor - 1 - \text{eq}(n_1, n_2)$. We have :*

- (a) $k + 3 \geq t_{NB}(G(n_1, n_2), k) \geq \begin{cases} k + 2 & \text{if } 4 \leq k \leq t_1 \\ k + 1 & \text{if } t_1 + 1 \leq k \leq D(G(n_1, n_2)) \end{cases}$
- (b) $t_{NB}(G(n_1, n_2), k) = k$ if $k = D(G(n_1, n_2))$

Proof. (b) is a direct application of the result of [4] concerning the usual broadcasting in 2-dimensional grids.

Now let us prove (a), and let us first prove the upper bound. For this, a k -NB is achievable in $k + 3$ rounds from any vertex v in $G(n_1, n_2)$. We suppose that n_1 and n_2 are “big enough”, that is we do not take the border into account. Let us prove by induction that it is possible to achieve a $k + 3$ rounds, with the property that the leftmost (resp. rightmost) vertex source, u_k (resp. v_k), is informed at round $k + 1$ at worst. Clearly, this property is satisfied for $k = 2$ and $k = 3$ (cf. Figure 3(middle) and (right)). Now, suppose it is true for a given $k \geq 2$, and let us show it still holds for $k + 1$. For this vertex v , different from u_k and v_k , lying at distance k from the source, by the information (that it obtained at round $k + 3$ at worst, by hypothesis) v is informed during round $k + 4$. The vertices that are “above” the source broadcast upwards, and the others downwards. Then 6 vertices remain uninformed : 3 are neighbors of u_k and 3 are neighbors of v_k . Note that u_k and v_k are informed at round $k + 1$ by induction hypothesis. Let u_k (resp. v_k) inform u_{k+1} on its left (resp. v_{k+1} on its right) during round $k + 2$, and inform its two still uninformed neighbors during rounds $k + 3$ and $k + 4$. This scheme is a $(k + 1)$ -NB scheme, running in $k + 3$ rounds, with the added property that u_{k+1} (resp. v_{k+1}) is informed at round

at worse. Hence the upper bound is proved by induction on k .

Now, depending on the values of n_1 and n_2 , the boundaries of the sometimes interfere ; however, it suffices to “cut” the figure where the boundaries appear, and one can see that it does not affect the scheme in the sense that all vertices within distance k are still reached. Thus $t_{NB}(G(n_1, n_2), k) \leq k + 3$ for $k \geq 2$ and any $n_1 \geq n_2 \geq 2$.

The lower bounds are obtained thanks to Proposition 1, $G(n_1, n_2)$ bipartite : when $k \leq t_1$, it is possible to find in $G(n_1, n_2)$ a vertex v such that $s_k(v) \geq k + 2$ vertices at distance k , thus $t_{NB}(G(n_1, n_2), k) \geq k + 2$. When $k \leq D(G(n_1, n_2)) - 1$, there exists at least one vertex v that has $s_k(v) \geq 2$ vertices at distance k , and thus $t_{NB}(G(n_1, n_2), k) \geq k + 1$.

4.2 2-Dimensional Tori

Proposition 7 For any integers n_1 and n_2 with $n_1 \geq n_2 \geq 3$, we have :

(a) $t_{NB}(TG(n_1, n_2), 1) = 3$;

(b) $k + 1 \leq t_{NB}(TG(n_1, n_2), k) \leq k + 3$ for any $2 \leq k \leq D(TG(n_1, n_2))$

(c) For any $k = D(TG(n_1, n_2))$,

$$t_{NB}(TG(n_1, n_2), k) = \begin{cases} k & \text{if } n_1 \text{ and } n_2 \text{ are even} \\ k + 1 & \text{otherwise} \end{cases}$$

Proof. In the case $k = 1$, at least 3 rounds are necessary by Property 1, there are 5 vertices to inform (source included). One can notice that the broadcast scheme in 3 rounds given in Figure 3(left) also applies for tori ; $TG(n_1, n_2)$ is a vertex-transitive, we conclude that $t_{NB}(TG(n_1, n_2), k) \leq 3$ and the equality holds.

In the case $2 \leq k \leq D(TG(n_1, n_2)) - 1$, we note that the upper bound given in Proposition 6 for grids also holds in tori. Indeed, similarly as for grids, the wrap-around in tori does not interfere in the ability of a vertex to broadcast at distance k . The lower bound of $k + 1$ in (b) derives from the fact that for every vertex $u \in V(TG(n_1, n_2))$, there exists at least two vertices lying at distance exactly k from u .

Finally, the result (c) comes from [4], where the usual broadcast time on 2-dimensional tori has been studied.

Proposition 8 For any integers $n_1 \geq n_2 \geq 4$ where n_1 and n_2 are both even, let $t_2 = \lfloor \frac{2(n_1 + n_2 - 1)}{5} \rfloor$. In that case, we have $t_{NB}(TG(n_1, n_2), k) \geq k + 2$ in the following cases :

(a) $n_1 \geq 4n_2 - 2$ and $k \leq 2n_2 - 2$;

(b) $n_1 = 4n_2 - 4$ and $k \leq 2n_2 - 3$;

(c) $n_1 \leq 4n_2 - 6$ and $k \leq t_2$.

Sketch of proof : We note that when n_1 and n_2 are both even, then $TG(n_1, n_2)$ is bipartite, and thus Proposition 1(b) applies in that case. Hence it suffices to consider the cases for which there exists in $TG(n_1, n_2)$ at least $k + 2$ vertices at distance exactly k from a given vertex u . The results given above describe exactly these cases. By Proposition 1(b), we conclude that in those cases, $t_{NB}(TG(n_1, n_2)) = k + 2$.

5 Conclusion

In this paper, we have studied for the first time the k -NB for general store-and-forward, 1-port, unit cost model. We have developed several algorithms and techniques that we have applied to several families of graphs, in order to obtain, as often as possible, exact results. In some other cases, we have obtained bounds that are close to the optimal (up to an additive constant never exceeding 2). Due to the length of the paper, we have not been able to state all our results. Notably, we are also able to determine good bounds for k -NB in d -dimensional grids and tori, triangular meshes and hypercubes. Also, as mentioned in the introduction, a similar study has been undertaken in the case of k -neighborhood gossiping. All these results will be detailed in the journal version of this paper.

The main drawback that we wish to state here is the lack of generalization of the methods we propose. We note that the same occurs in previous papers on k -NB, either 1-NB or 1-NG in specific families of graphs [2, 1, 6, 9]. Non surprisingly, some difficulties also occur when the studied topology is not vertex-transitive.

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Guillaume Fertin is with IRIN, Université de Nantes - 2 rue de la Houssinière, 44322 Nantes Cedex 3 - France. E-mail: fertin@irin.univ-nantes.fr

André Raspaud is with LaBRI, Université Bordeaux 1 - 351 cours de la Libération, 33000 Talence Cedex - France. E-mail: raspaud@labri.fr