k-Neighborhood Broadcasting*

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Abstract

Broadcasting refers to the task whereby a node in a network, knowin
piece of information, must transmit it to all the other nodes. In this pap
we consider a generalized form of broadcasting, that we call k-neighborho
broadcasting. It consists in the following : a node u in the network has
send its information to all the nodes which are at distance less than or eq
to k from u.

We study k-neighborhood broadcasting (or k-NB for short) in paths, tre
cycles, 2-dimensional grids and 2-dimensional tori under the store and f
ward, 1-port, unit cost model. For most of these families, we give the optin
k-NB time ; if not, the optimal k-NB time is given within an additive consta
never exceeding 2.

Keywords

Broadcasting, communications, networks.

1 Introduction and Preliminaries

We model a network by a graph, where the vertices and the edges r
respectively the processors and the communication links of the network
networks, broadcasting consists, for a source node u of the network holdi
information, in sending this information to all the other nodes in the
Broadcasting has been widely studied in the previous years, and has give
several books and articles. We refer to [10, 5, 3, 11] for comprehensive
on this problem.

In this paper, we consider a generalized version of this problem, tha
k-neighborhood broadcasting (or k-NB for short) ; here, broadcasting
realized at distance k in the graph G, $1 \leq k \leq D(G)$, where $D(G)$ denote
ameter of G. More formally, k-NB from u consists in sending u's inform

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all the nodes which are within distance k from u.

The model we use here is *store-and-forward* (messages go to their de
by transiting in the intermediate nodes ; that is, communications always ta
between neighbors), *1-port* (at each given time, a node can communicat
most one of its neighbors) and *unit cost* (a communication between two n
takes one time unit, or *round*).

When $k = D(G)$, then the k-NB problem is the usual broadcasting p
and as mentioned earlier, a wide literature on the subject exists. We note th
has also been studied in the particular case when $k = 1$ in two different
of graphs, namely hypercubes [2, 1] and star graphs [9].

We note that even in the case $k = 1$, the time to achieve 1-NB in a gr
not necessarily simply equal to the maximum degree $\Delta(G)$ of the graph
this, consider the wheel graph $W_{1,n}$ that consists of a cycle of order n, C_n,
we add a "center" vertex c, connected to all the vertices of C_n. In that cas
be seen that the time to achieve 1-NB from any vertex different from c
to 2, and the time to achieve 1-NB from c is equal to $\left\lceil \frac{1+\sqrt{1+4(n-1)}}{2} \right\rceil$, th
degree of c, $d(c)$, satisfies $d(c) = \Delta(W_{1,n}) = n$.

Let us also mention that another usual type of communication in n
gossiping (where every node must send its information to every other n
be similarly generalized to k-neighborhood gossiping (or k-NG). In the
lar case $k = 1$, the 1-NG problem has been studied in hypercubes [6]
graphs [9] ; for general k, a study of k-NG, similar to the one presente
paper, has been undertaken in [7].

In this paper, we give bounds on the optimal time for k-NB in the fo
families of graphs : paths, trees, cycles, 2-dimensional grids and tori. In
those families, we determine the optimal k-NB time ; if not, we give bo
the optimal k-NB time, which differ by an additive constant not exceed
Section 2, we give some necessary definitions and determine some gene
erties about the minimum time to achieve k-NB in a graph G. In Sections
we study k-NB in paths, trees of maximum degree Δ and diameter D, c
dimensional grids and 2-dimensional tori.

2 Generalities

We denote by $t_{NB}(G,u,k)$ the minimum time to achieve k-NB from
in the graph G under our model. We call the *k-neighborhood broadcas*
G, or $t_{NB}(G,k)$, the maximum $t_{NB}(G,u,k)$ over all $u \in V(G)$. Let also
be the number of vertices which are at distance less than or equal to k fr
included), and $N_k(G) = \max_{u \in V(G)} N_k(G,u)$. We then have the following p

Property 1 *For any graph G and for any $1 \leq k \leq D(G)$,*

$$t_{NB}(G,k) \geq \max\{k, \lceil \log_2(N_k(G)) \rceil\}$$

Proof. Any vertex u in a graph G which has to inform t vertices (u in
needs at least $\lceil \log_2(t) \rceil$ time units in our model (at each time unit, the nu
informed nodes can at most double). Since in k-NB, there exists at least o
u in G that must inform $N_k(G)$ vertices, we then get $t_{NB}(G,k) \geq \lceil \log_2($
Finally, it is easy to see that $t_{NB}(G,k) \geq k$: indeed, $k \leq D(G)$, hence ther
exists two vertices u and v at distance k. Since u needs to broadcast its info
to v, we have $t_{NB}(G,u,k) \geq k$, and thus $t_{NB}(G,k) \geq k$.

Property 2 *Let G be a bipartite graph and u a vertex of G. For any $1 \leq k$
and any $p \geq 0$, we denote by $N(k,p)$ the maximum number of vertices at*
exactly *k from u that can be informed by u within $k+p$ rounds. In th*
$N(k,p) \leq \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} \binom{k+p}{k+2i}$.

Proof. Let $u \in V(G)$. If u is to inform a vertex v at distance exactly
p rounds, it can either use a shortest path, or it can reach v via a lon
However, since G is a bipartite graph, the latter only happens when the l
this path is of the same parity as k. Thus, in each case, a path of lengt
$0 \leq i \leq \lfloor \frac{p}{2} \rfloor$ can be used to reach a vertex at distance k. In order to d
how many such paths are possible, it suffices to note that the number of b
sequences from u to v along a path of length $k+2i$, using rounds be
and $k+p$, is equal to the binomial $\binom{k+p}{k+2i}$. Summing this term ov
possible values of i (i.e., $0 \leq i \leq \lfloor \frac{p}{2} \rfloor$), we get the result.

Thanks to the above property, and thanks to a well-known observatio
deduce the following proposition, that we will use several times in the re
paper.

Proposition 1 *For any graph G, let u be a vertex of G. For any $1 \leq k \leq$
$s_k(u)$ be the number of vertices at distance* exactly *k from u. Then :*

(a) If $s_k(u) \geq 2$, $t_{NB}(G,k) \geq k+1$;

(b) If G is bipartite and $s_k(u) \geq k+2$, $t_{NB}(G,k) \geq k+2$.

In the following, we denote by $G \square H$ the cartesian product of grap
H.

Proposition 2 *For any graphs G and H, and for any $1 \leq k \leq \min\{D(G$
$D(G \square H)\}$, $t_{NB}(G \square H, k) \leq t_{NB}(G,k) + t_{NB}(H,k)$.*

Proof. The broadcast scheme we propose here from any vertex $(u,v) \in V$
is the following : achieve a k-NB from (u,v) to all the vertices of the for
that are in the copy of G containing (u,v). Then, achieve a k-NB fron
the vertices (u',v) reached by the first part of the scheme, to every verte
within distance k from (u',v), that lies in the copy of H containing (u',v).
this scheme respect the 1-port model, and is achieved in time $t_{NB}(G,k)+t_{l}$
Moreover, this is a k-NB scheme : indeed, it can be easily seen that ever
within distance k from (u,v) is informed.

3 Paths, Trees and Cycles

3.1 Paths

Theorem 1 *Let P_n be the path of order n. For any n and $1 \leq k \leq n-1$,*

$$t_{NB}(P_n,k) = \begin{cases} k & \text{if } k > \lfloor \frac{n-1}{2} \rfloor \\ k+1 & \text{otherwise} \end{cases}$$

Proof. By Property 1, we know that for any $1 \leq k \leq D(P_n)$, that is
$1 \leq k \leq n-1$, we have $t_{NB}(P_n,k) \geq k$. Moreover, let us show that when
$\lfloor \frac{n-1}{2} \rfloor$, there exists at least one vertex u that has to inform 2 vertices at
k, which implies by Proposition 1(a) that $t_{NB}(P_n,u,k) \geq k+1$. For thi
denote by $u_1, u_2 \ldots u_n$ the vertices of P_n, from left to right, and let us
vertex $u_{\lfloor \frac{n-1}{2} \rfloor +1}$. It has $\lfloor \frac{n-1}{2} \rfloor$ vertices on "its left" (that is, vertices denote
$1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$), and $n-1-\lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n-1}{2} \rceil \geq \lfloor \frac{n-1}{2} \rfloor$ vertices on "its right"
for any $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $t_{NB}(P_n,k) \geq k+1$. Note also that for any $\lfloor \frac{n-1}{2} \rfloor < k$
then for any $u_i \in V(P_n)$, there is at most one vertex at distance k from u_i.

Now, for any vertex u_i in P_n and every $1 \leq k \leq n-1$, the broadcast
is as follows : if u_i has two vertices at distance k, then broadcast along
(resp. left) edges from round 1 to round k (resp. from round 2 to roun
If u_j is the only vertex at distance k from u_i, then if $j > i$ (resp. $j < i$) b
from round 1 to round k on the right (resp. left) edges, and from round 2
$|j-i|+1$ on the left (resp. right) ones. If there is no vertex at distance
u_i then broadcast to the left from round 1, and to the right from round
the cases, it can be seen that the broadcast scheme we propose here m
appropriate lower bound.

Non surprisingly, in the case $k = D(P_n) = n-1$, we obtain the same
in [4] concerning the usual broadcast problem, that is $t_{NB}(P_n, n-1) = n$

3.2 Trees of maximum degree Δ and diameter D

In this section, we determine t_{NB} for the family $\mathcal{T}_{\Delta,D}$ of trees with m
degree Δ and diameter D. We denote by $t_{NB}(\mathcal{T}_{\Delta,D},k)$ the maximum k-
over all the trees $T \in \mathcal{T}_{\Delta,D}$.

Definition 1 ($\mathbb{T}_0(\Delta,D)$ and $\mathbb{T}_1(\Delta,D)$) *For any fixed Δ and even D, we d* *$\mathbb{T}_0(\Delta,D)$ the unique tree rooted at r such that the leaves lie at distance e* *from r, and such that all the vertices (except the leaves) are of degree Δ.*

For any odd $D = 2D'+1$, $\mathbb{T}_1(\Delta,D)$ can be obtained from 2 copies T *of $\mathbb{T}_0(\Delta,2D')$, where we delete a branch incident to the root r_i in each c* *$(i \in \{1,2\})$, and where we then connect the two roots r_1 and r_2 by an edg*

Examples of such trees are given in Figure 1, where we show resp $\mathbb{T}_0(4,4)$ and $\mathbb{T}_1(4,5)$.

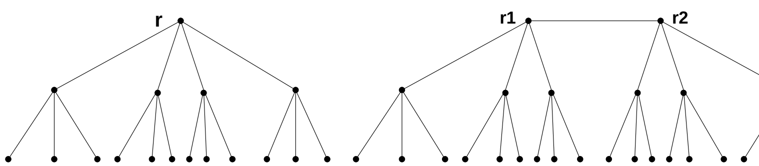

Figure 1: $\mathbb{T}_0(4,4)$ (left) and $\mathbb{T}_1(4,5)$ (right).

Clearly, $\mathbb{T}_0(\Delta,D) \in \mathcal{T}_{\Delta,D}$ (resp. $\mathbb{T}_1(\Delta,D) \in \mathcal{T}_{\Delta,D}$) for any fixed Δ an (resp. odd D). More generally, for any fixed Δ, it is easy to see that for a D, any tree $T \in \mathcal{T}_{\Delta,D}$ is a subtree of $\mathbb{T}_0$, and for any odd D, any tree $T \in$ a subtree of $\mathbb{T}_1$.

We also note that for odd D, $\mathbb{T}_1(\Delta,D)$ can be seen as a copy of $\mathbb{T}_0(\Delta,$ which half of the leaves (in either the left or right subtree of r) has been r

Lemma 1 *For any fixed Δ and D, and for any $1 \leq k \leq \lfloor \frac{D}{2} \rfloor$:*

(a) if D is even, $t_{NB}(\mathbb{T}_0(\Delta,D),k) \geq k\cdot(\Delta-1)+1$;

(b) if D is odd, $t_{NB}(\mathbb{T}_1(\Delta,D),k) \geq k\cdot(\Delta-1)+1$.

Proof. Suppose first D is even, and let us show that for any $1 \leq$ $t_{NB}(\mathbb{T}_0(\Delta,D),r,k) \geq k\cdot(\Delta-1)+1$, where r is the root of $\mathbb{T}_0(\Delta,D)$. This by induction on k. When $k=1$, it takes time Δ for r to inform all vertic tance k. Now suppose that $t_{NB}(\mathbb{T}_0(\Delta,D),r,k) \geq k\cdot(\Delta-1)+1$ for a fixe and let us show that this implies $t_{NB}(\mathbb{T}_0(\Delta,D),r,k+1) \geq (k+1)\cdot(\Delta-$ provided that $k+1 \leq \frac{D}{2}$. There exists a vertex v in $\mathbb{T}_0(\Delta,D)$ at distance which is informed after round $k\cdot(\Delta-1)+1$. Moreover, since $k+1 \leq$ $\Delta-1$ sons in $\mathbb{T}_0(\Delta,D)$ which are at distance $k+1$ from r. Hence it wi

least $\Delta-1$ more rounds for v to inform its $\Delta-1$ sons in $\mathbb{T}_0(\Delta,D)$, and
have $t_{NB}(\mathbb{T}_0(\Delta,D),r,k+1) \geq k\cdot(\Delta-1)+1+\Delta-1$. Thus the result is p
induction on k.

Now suppose D is odd. In that case, for any $k \leq \frac{D-1}{2}$, it is easy to see
subtree of $\mathbb{T}_1(\Delta,D)$ induced by the vertices at distance less than or equal t
r_1 is isomorphic to $\mathbb{T}_0(\Delta,2k)$, with $r=r_1$. Thus, since by Lemma 1(a), v
that $t_{NB}(\mathbb{T}_0(\Delta,2k),r,k) \geq k\cdot(\Delta-1)+1$, we get $t_{NB}(\mathbb{T}_1(\Delta,D),r,k) \geq k\cdot(\Delta$
Hence $t_{NB}(\mathbb{T}_1(\Delta,D),k) \geq k\cdot(\Delta-1)+1$, and the result is proved.

Lemma 2 *For any fixed* Δ, D *and* $1 \leq k \leq \lfloor \frac{D}{2} \rfloor$ *:*

(a) if D is even, $t_{NB}(\mathbb{T}_0(\Delta,D),k) \leq k\cdot(\Delta-1)+1$ *;*

(b) if D is odd, $t_{NB}(\mathbb{T}_1(\Delta,D),k) \leq k\cdot(\Delta-1)+1$.

Proof. First suppose D is even. Let v be any vertex of $\mathbb{T}_0(\Delta,D)$. The m
here is to note that the subtree of $\mathbb{T}_0(\Delta,D)$ induced by the vertices th
distance less than or equal to k from v is itself a subtree T of $T_v = \mathbb{T}_0(\Delta,2k$
T_v is rooted at v (cf. Figure 2, where $\Delta=4$, $D=6$ and $k=3$). However, i
consequently in T, we know that we can achieve k-NB from v within $k\cdot(\Delta$
rounds. Hence, for any vertex $v\in\mathbb{T}_0(\Delta,D)$, we have $t_{NB}(\mathbb{T}_0(\Delta,D),v,k) \leq$
$1)+1$. This proves the result when D is even.

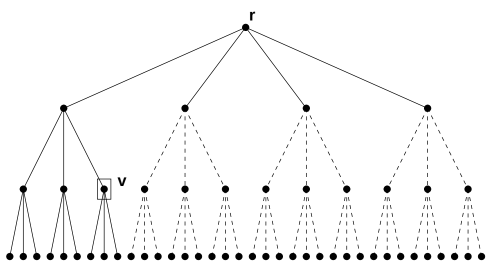

Figure 2: In $\mathbb{T}_0(4,6)$, we construct T rooted at v, with $k=3$.

Similarly, when D is odd, for any $v\in\mathbb{T}_1(\Delta,D)$, the subtree T' ind
the set of vertices that lie within distance k of v in $\mathbb{T}_1(\Delta,D)$ is itself a su
$T'_v = \mathbb{T}_0(\Delta,2k)$, where T'_v is rooted at v. Since we know that in T'_v (and th
it is possible to achieve k-NB from v within $k\cdot(\Delta-1)+1$ rounds, we th
$t_{NB}(\mathbb{T}_1(\Delta,D),v,k) \leq k\cdot(\Delta-1)+1$ for any v in $\mathbb{T}_1(\Delta,D)$, and thus the
also proved for odd D.

Lemma 3 *For any fixed* Δ, D *and* $\lfloor \frac{D}{2} \rfloor + 1 \leq k \leq D$ *:*

(a) if D is even, $t_{NB}(\mathbb{T}_0(\Delta,D),k) \geq \frac{D}{2}\cdot(\Delta-2)+k$ *;*

(b) if D is odd, $t_{NB}(\mathbb{T}_1(\Delta,D),k) \geq \lfloor \frac{D}{2} \rfloor \cdot (\Delta-2)+k$.

Proof. Suppose first D is even, and take any vertex v that lies at distan from the root r in $\mathbb{T}_0(\Delta,D)$ (such a vertex exists, since $k \leq D$). Actually, sen such that it must broadcast to all the other vertices in the tree. Let r_v b of r that is on the path from v to r, and let T' be the subtree of $\mathbb{T}_0(\Delta,D)$ r_v, and obtained by deleting the edge rr_v. Any vertex not in T' must be i via r ; that is, r must inform all the vertices of $T_r = \mathbb{T}_0(\Delta,D) - T'$. Mo cannot be informed before round $R_1 = k - \frac{D}{2}$. Now, in order for r to infor vertices of T_r, it can be seen that we need at least $R_2 = \frac{D}{2}(\Delta-1)$ rounds. S R_1 and R_2, we get $t_{NB}(\mathbb{T}_0(\Delta,D),v,k) \geq \frac{D}{2} \cdot (\Delta-2)+k$.

Now let us assume D is odd. In that case, we use the same kind of ar let T_1 (resp. T_2) be the subtree of $\mathbb{T}_1(\Delta,D)$ that contains r_1 (resp. r_2) th tained by removing the edge r_1r_2, and let v be a vertex lying at distance $\lfloor$ r_1 in T_1. v has been chosen so that it must broadcast to all the vertices in $\mathbb{T}$ In particular all the vertices of T_2 must be informed via r_1 and r_2. Moreov denote by r_1' the son of r_1 that lies on the path from v to r_1, and if we d T' the subtree containing r_1' obtained by deleting the edge $r_1'r_1$, all the ve $T_1 - T'$ must also be informed via r_1. Whatever the strategy used, and in lar whatever round is used on edge r_1r_2, it can be shown that there exists $T - T'$ that cannot be informed by u in less than $k - \lfloor \frac{D}{2} \rfloor + (\Delta-1)\lfloor \frac{D}{2} \rfloor$ ro other words, we have $t_{NB}(\mathbb{T}_1(\Delta,D),v,k) \geq \lfloor \frac{D}{2} \rfloor \cdot (\Delta-2)+k$ and thus the proved.

Lemma 4 *For any fixed Δ, D and $\lfloor \frac{D}{2} \rfloor + 1 \leq k \leq D$:*

(a) if D is even, $t_{NB}(\mathbb{T}_0(\Delta,D),k) \leq \frac{D}{2} \cdot (\Delta-2)+k$;

(b) if D is odd, $t_{NB}(\mathbb{T}_1(\Delta,D),k) \leq \lfloor \frac{D}{2} \rfloor \cdot (\Delta-2)+k$.

Proof. Suppose first D is even, and let $\frac{D}{2}+1 \leq k \leq D$. First, we know t vertex in $\mathbb{T}_0(\Delta,D)$ is at distance at most $\frac{D}{2}$ from r. Hence, any k-NB fro $k \geq \frac{D}{2}+1$ is equivalent to a $\frac{D}{2}$-NB from r, and thus takes at most $\frac{D}{2}(\Delta$ rounds. Since $k \geq \frac{D}{2}+1$, we see that we always have $\frac{D}{2}(\Delta-1)+1 \leq k+\frac{D}{2}$ In other words, any k-NB from r always satisfies the inequality (a) of L above.

Now, for any vertex $v \neq r$ in $\mathbb{T}_0(\Delta,D)$, let $dist(v,r)$ be its distance root r, and let the *local root* lr be defined as follows : $lr = r$ if $dist(v,r)$ and lr is the vertex lying at distance $k - \frac{D}{2}$ from v on the path fror otherwise. Two cases then arise :

(a) $lr = r$. In that case, this means that v must broadcast to all the in the tree. Let v inform r using rounds $1,2,\ldots,dist(v,r)$. Let us deno the son of r that lies on the path from v to r, and by T' the tree roo that is obtained by removing the edge rr'. Now, r informs all the ve $\mathbb{T}_0(\Delta,D)-T'$ starting from round $dist(v,r)+1$. This is achieved using $\frac{D}{2}(\Delta-2)+k$ rounds on the whole. During that time, r' broadcasts in T' ($\frac{D}{2}-1$), with the constraint that the path from v to r is already allow rounds. However, this can be achieved within round $\frac{D}{2}(\Delta-2)+k-$ globally, the broadcast scheme uses at most $\frac{D}{2}(\Delta-2)+k$ rounds.

(b) $lr \neq r$. In that case, v does not need to broadcast to all the ve the tree : its k-neighborhood does not contain every vertex in the tree. H using the same notation as previously, it must broadcast to all the vertic and to all the vertices lying at distance less than or equal to $\frac{D}{2}$ from lr in $\mathbb{T}_0(\Delta,D)-T'$. One can also see that the depth of T' is less than or equ hence, globally the broadcast scheme described in the previous case rem same, except that lr is substituted to r. The maximum number of round here is also $\frac{D}{2}(\Delta-2)+k$, and thus the result is proved.

When D is odd, we use a very similar kind of argument to sh in $\mathbb{T}_1(\Delta,D)$, k-NB can be achieved from any vertex v in $\lfloor\frac{D}{2}\rfloor\cdot(\Delta-2)+$
□

Theorem 2 $t_{NB}(\mathcal{T}_{\Delta,D},k)=\begin{cases} k\cdot(\Delta-1)+1 & if\ 1\leq k\leq\lfloor\frac{D}{2}\rfloor \\ \lfloor\frac{D}{2}\rfloor\cdot(\Delta-2)+k & if\ \lfloor\frac{D}{2}\rfloor<k\leq D\end{cases}$

Proof. We know that any tree $T\in\mathcal{T}_{\Delta,D}$ is a subtree of $\mathbb{T}_0(\Delta,D)$ wh even (resp. a subtree of $\mathbb{T}_1(\Delta,D)$ when D is odd). Thus we have t_{NB} $t_{NB}(\mathbb{T}_0(\Delta,D),k)$ for any $T\in\mathcal{T}_{\Delta,D}$ with even D, and $t_{NB}(T,k)\leq t_{NB}(\mathbb{T}_1($ for any $T\in\mathcal{T}_{\Delta,D}$ with odd D.

Moreover, by Lemmas 1 to 4 we have :

- If D is even, then $t_{NB}(\mathbb{T}_0(\Delta,D),k)=\begin{cases} k\cdot(\Delta-1)+1 & \text{when } 1\leq \\ \frac{D}{2}\cdot(\Delta-2)+k & \text{when } \frac{D}{2}<\end{cases}$
- If D is odd, then
$$t_{NB}(\mathbb{T}_1(\Delta,D),k)=\begin{cases} k\cdot(\Delta-1)+1 & \text{when } 1\leq k\leq\frac{D-1}{2} \\ \frac{D-1}{2}\cdot(\Delta-2)+k & \text{when } \frac{D-1}{2}<k\leq D\end{cases}$$

Hence the result.

Remark: We note that Theorem 1 is a particular case of Theorem Indeed, if we suppose $\Delta=2$, $\mathcal{T}_{\Delta,D}$ only contains one tree, more precisely P_{D+1}. Putting now $n=D+1$ gives directly Theorem 1.

We also note that there exists several results concerning the (usual) b
time in some specific types of trees, such as complete Δ-ary trees of dep
for instance [3]). However, these results cannot be compared to ours,
complete Δ-ary trees, every vertex not being a leaf has Δ sons. This mean
complete Δ-ary trees, the root is of degree Δ, and every other vertex no
leaf is of degree $\Delta+1$; in our case, every vertex not a leaf (root includ
degree Δ. This is not to mention that complete Δ-ary trees of depth h are
diameter $D = 2h$, and that in our case we do not restrict D to be even.

3.3 Cycles

Theorem 3 *Let C_n be the cycle of order n. For any $n \geq 3$ and any $1 \leq k$*

$$t_{NB}(C_n,k) = \begin{cases} k & \text{if } n \text{ is even and } k = \frac{n}{2} \\ k+1 & \text{otherwise} \end{cases}$$

Proof. For any n, C_n is vertex-transitive, thus we can focus on broa
from a given vertex $v \in V(C_n)$. Suppose first n is odd. In that case,
$1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, there are two vertices at distance k from v, and thus by
tion 1(a) $t_{NB}(C_n,k) \geq k+1$. When n is even, there are also 2 vertices at d
from v, except in the case $k = \frac{n}{2}$, where there is only one such vertex. Th
n is even, $t_{NB}(C_n,k) \geq k+1$ when $k \neq \frac{n}{2}$ by Proposition 1(a) and $t_{NB}(C$
otherwise.

Now let us show that these bounds are tight. For this, let v broadcast
mation at distance k in C_n as follows : at round 1, v sends its informatio
of its neighbor, and in the following rounds, every newly informed vert
the information to its still uninformed neighbor. Following this scheme,
seen that any k-NB in C_n is achieved in $k+1$ rounds, except in the cas
and $k = \frac{n}{2}$, where k rounds suffice (since, as mentioned above, for any v
there exists only one vertex v' at distance k from v).

Here again, we note that in the case $k = D(C_n) = \lfloor \frac{n}{2} \rfloor$, we obtain
known result concerning (usual) broadcasting in cycles ; that is, $t_{NB}(C_n$
$\lceil \frac{n}{2} \rceil$.

4 2-Dimensional Grids and Tori

In the following, $G(n_1,n_2)$ (resp. $TG(n_1,n_2)$) will denote the 2-dim
grid (resp. torus) with n_1 rows and n_2 columns. Due to symmetries, we
ways suppose $n_1 \geq n_2$. We also denote here by $D(G(n_1,n_2))$ the dia
the 2-dimensional grid $G(n_1,n_2)$, and by $D(TG(n_1,n_2))$ the diameter o
dimensional torus $TG(n_1,n_2)$. Thus we have $D(G(n_1,n_2)) = n_1 + n_2$

$D(TG(n_1,n_2)) = \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor$.

4.1 2-Dimensional Grids

Proposition 3 (1-NB in 2-dim. Grids) *For any integers n_1, n_2 with $n_1 \geq$*

$$t_{NB}(G(n_1,n_2),1) = \begin{cases} 2 & \textit{if } n_1 = n_2 = 2 \\ 3 & \textit{otherwise} \end{cases}$$

Proof. When $n_1 = n_2 = 2$, it is easy to see that 2 rounds are neces sufficient for any vertex to broadcast at distance 1. In all the other cas exists in the grid at least one vertex v for which the number of vertices at 1 from v is greater than or equal to 3. Since $G(n_1,n_2)$ is a bipartite grap to Proposition 1(b) we conclude that $t_{NB}(G(n_1,n_2),1) \geq 3$. However, it i see that 3 rounds are sufficient in that case (cf. for this Figure 3(left)).

Proposition 4 (2-NB in 2-dim. Grids) *For any integers n_1, n_2 with $n_1 \geq$*

$$t_{NB}(G(n_1,n_2),2) = \begin{cases} 2 & \textit{if } n_1 = n_2 = 2 \\ 3 & \textit{if } n_1 \in \{3,4\} \textit{ and } n_2 = 2 \\ 4 & \textit{if } n_1 \geq 5 \textit{ and } n_2 = 2 \textbf{ or } \textit{if } n_2 \in \{3,4\} \\ 5 & \textit{otherwise (that is, if } n_1 \geq n_2 \geq 5\textit{)} \end{cases}$$

Sketch of Proof : The proof relies here on detailing all the possible case that, for fixed n_2, for any $n_1 \geq 5$, the result is the same as for $n_1 = 5 =$ For any $n_2 \leq 4$, by a rather tedious case by case analysis, the above re be deduced. If $n_1 \geq n_2 \geq 5$, we note that there exists a vertex v in $G(n_1$ must inform $4k = 8$ vertices at distance $k = 2$. By Property 2, if we are $k+2 = 4$ rounds, then v can inform at most $\frac{(k+2)(k+1)}{2} + 1 = 7$ vertices at 2. Thus, in that case $t_{NB}(G(n_1,n_2),2) \geq 5$; it is easy to see that 5 round for any vertex in $G(n_1,n_2)$ (with $n_1 \geq n_2 \geq 5$) to broadcast its informatio distance 2 (cf. for instance Figure 3(middle)).

Proposition 5 (3-NB in 2-dim. Grids) *For any integers $n_1 \geq 3$ and n_2 $n_1 \geq n_2$,*

$$t_{NB}(G(n_1,n_2),3) = \begin{cases} 3 & \textit{if } n_1 = 3 \textit{ and } n_2 = 2 \\ 4 & \textit{if } n_1 \in \{4,5,6\} \textit{ and } n_2 = 2 \\ & \textbf{or } \textit{if } n_1 \in \{3,4\} \textit{ and } n_2 = 3 \textbf{ or } \textit{if } n_1 = n_2 = 4 \\ 5 & \textit{if } n_1 \geq 7 \textit{ and } n_2 = 2 \textbf{ or } \textit{if } n_1 \geq 5 \textit{ and } n_2 \in \{3, \\ & \textbf{or } \textit{if } n_1 \in \{5,6\} \textit{ and } n_2 = 5 \\ 6 & \textit{otherwise} \end{cases}$$

Sketch of Proof : Here again, we detail all the possible cases up to $G(7,7$ over, when $n_1 \geq n_2 \geq 7$, by Property 2 we can show that $k+3 = 6$ rounds essary to achieve a k-NB from any of the vertices of $G(n_1,n_2)$; as in the

case, it is easy to show that 6 rounds are enough (cf. for instance Figure 3
□

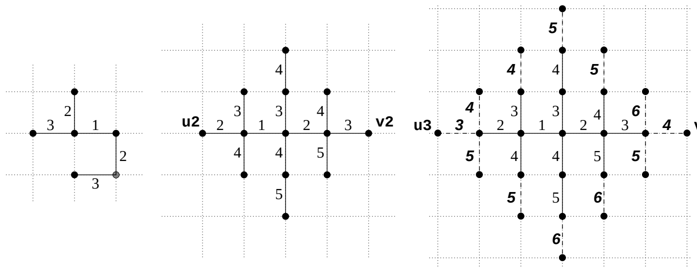

Figure 3: 1-NB (left), 2-NB (middle) and 3-NB (right) in $G(n_1, n_2$

Proposition 6 (k-NB in 2-dim. Grids) *For any integers n_1, n_2 with $n_1 \geq$ and $n_1 + n_2 \geq 6$, let $\mathrm{eq}(n_1, n_2) = 1$ if $n_1 = n_2$ and 0 otherwise. Let* t $\left\lfloor \frac{2n_2+n_1}{3} \right\rfloor - 1 - \mathrm{eq}(n_1, n_2)$. *We have :*

$$(a)\;\; k+3 \geq t_{NB}(G(n_1,n_2),k) \geq \begin{cases} k+2 & \text{if } 4 \leq k \leq t_1 \\ k+1 & \text{if } t_1+1 \leq k \leq D(G(n_1,n_2)) \end{cases}$$

$$(b)\;\; t_{NB}(G(n_1,n_2),k) = k \text{ if } k = D(G(n_1,n_2))$$

Proof. (b) is a direct application of the result of [4] concerning the usua casting in 2-dimensional grids.

Now let us prove (a), and let us first prove the upper bound. For this, a k-NB achievable in $k+3$ rounds from any vertex v in $G(n_1, n_2)$. We suppose that n_1 and n_2 are “big enough”, that is we do not take the bo into account. Let us prove by induction that it is possible to achieve a $k+3$ rounds, with the property that the leftmost (resp. rightmost) vertex source, u_k (resp. v_k), is informed at round $k+1$ at worse. Clearly, this pr satisfied for $k = 2$ and $k = 3$ (cf. Figure 3(middle) and (right)). Now, sup is true for a given $k \geq 2$, and let us show it still holds for $k+1$. For this vertex v, different from u_k and v_k, lying at distance k from the source, b the information (that it obtained at round $k+3$ at worse, by hypothesis) v during round $k+4$. The vertices that are “above” the source broadcast u the others downwards. Then 6 vertices remain uninformed : 3 are neighbo and 3 are neighbors of v_k. Note that u_k and v_k are informed at round $k+1$ by induction hypothesis. Let u_k (resp. v_k) inform u_{k+1} on its left (resp. v_k right) during round $k+2$, and inform its two still uninformed neighbor rounds $k+3$ and $k+4$. This scheme is a $(k+1)$-NB scheme, running rounds, with the added property that u_{k+1} (resp. v_{k+1}) is informed at rou at worse. Hence the upper bound is proved by induction on k.

Now, depending on the values of n_1 and n_2, the boundaries of the sometimes interfere ; however, it suffices to "cut" the figure where the bo appear, and one can see that it does not affect the scheme in the sense th vertices within distance k are still reached. Thus $t_{NB}(G(n_1,n_2),k) \leq k+3$ $k \geq 2$ and any $n_1 \geq n_2 \geq 2$.

The lower bounds are obtained thanks to Proposition 1, $G(n_1,n_2)$ b partite : when $k \leq t_1$, it is possible to find in $G(n_1,n_2)$ a vertex v $s_k(v) \geq k+2$ vertices at distance k, thus $t_{NB}(G(n_1,n_2),k) \geq k+2$. When $k \leq D(G(n_1,n_2))-1$, there exists at least one vertex v that has $s_k(v) \geq 2$ at distance k, and thus $t_{NB}(G(n_1,n_2),k) \geq k+1$.

4.2 2-Dimensional Tori

Proposition 7 *For any integers n_1 and n_2 with $n_1 \geq n_2 \geq 3$, we have :*

(a) $t_{NB}(TG(n_1,n_2),1) = 3$ *;*

(b) $k+1 \leq t_{NB}(TG(n_1,n_2),k) \leq k+3$ *for any* $2 \leq k \leq D(TG(n_1,n_2))$

(c) For any $k = D(TG(n_1,n_2))$,

$$t_{NB}(TG(n_1,n_2),k) = \begin{cases} k & \text{if } n_1 \text{ and } n_2 \text{ are even} \\ k+1 & \text{otherwise} \end{cases}$$

Proof. In the case $k=1$, at least 3 rounds are necessary by Property there are 5 vertices to inform (source included). One can notice that t scheme in 3 rounds given in Figure 3(left) also applies for tori ; $TG(n_1,n$ vertex-transitive, we conclude that $t_{NB}(TG(n_1,n_2),k) \leq 3$ and the equali

In the case $2 \leq k \leq D(TG(n_1,n_2))-1$, we note that the upper bound given in Proposition 6 for grids also holds in tori. Indeed, similarly a boundaries in grids, the wrap-around in tori does not interfere in the a a vertex to broadcast at distance k. The lower bound of $k+1$ in (b) deri the fact that for every vertex $u \in V(TG(n_1,n_2))$, there exists at least two lying at distance exactly k from u.

Finally, the result (c) comes from [4], where the usual broadcasti dimensional tori has been studied.

Proposition 8 *For any integers $n_1 \geq n_2 \geq 4$ where n_1 and n_2 are both* $t_2 = \lfloor \frac{2(n_1+n_2-1)}{5} \rfloor$*. In that case, we have* $t_{NB}(TG(n_1,n_2),k) \geq k+2$ *in the f cases :*

(a) $n_1 \geq 4n_2-2$ *and* $k \leq 2n_2-2$ *;*

(b) $n_1 = 4n_2-4$ *and* $k \leq 2n_2-3$ *;*

(c) $n_1 \leq 4n_2 - 6$ *and* $k \leq t_2$.

Sketch of proof : We note that when n_1 and n_2 are both even, then $TG($
bipartite, and thus Proposition 1(b) applies in that case. Hence it suffices
the cases for which there exists in $TG(n_1, n_2)$ at least $k+2$ vertices at
exactly k from a given vertex u. The results given above describe exac
cases. By Proposition 1(b), we conclude that in those cases, $t_{NB}(TG(n_1,$
$k+2$.

5 Conclusion

In this paper, we have studied for the first time the k-NB for genera
store-and-forward, 1-port, unit cost model. We have developed several
and techniques that we have applied to several families of graphs, in
obtain, as often as possible, exact results. In some other cases, we ha
bounds that are close to the optimal (up to an additive constant never e
2). Due to the length of the paper, we have not been able to state all ou
Notably, we are also able to determine good bounds for k-NB in d-dim
grids and tori, triangular meshes and hypercubes. Also, as mentioned
similar study has been undertaken in the case of k-neighborhood gossi
All these results will be detailed in the journal version of this paper.

The main drawback that we wish to state here is the lack of generali
methods we propose. We note that the same occurs in previous papers
either 1-NB or 1-NG in specific families of graphs [2, 1, 6, 9]. Non sur
some difficulties also occur when the studied topology is not vertex-tran

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