Neighborhood Communications in Networks

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Abstract

Broadcasting (resp. gossiping) refers to the task whereby a node (resp. every node) in a network, knowing a piece of information, must transmit it to all the other nodes. We consider here a generalized form of broadcasting and gossiping, that we call respectively k-neighborhood broadcasting and k-neighborhood gossiping (k-NB and k-NG, for short). It consists in the following : a node u (resp. every node u) in the network has to send its information to all the nodes which are at distance less than or equal to k from u. We study k-NB and k-NG in paths, trees, cycles, 2-dimensional grids and 2-dimensional tori under the store-and-forward, 1-port, unit cost model. For most of these families, we give the optimal k-NB (resp. k-NG) time ; if not, the optimal k-NB (resp. k-NG) time is given within an additive (resp. multiplicative) constant never exceeding 2.

1 Introduction and Generalities

We model a network by a graph, where the vertices and the edges represent respectively the processors and the communication links of the network. In such networks, broadcasting (resp. gossiping) consists, for a source node (resp. for every node) in the network, in sending its information to all the other nodes. We consider a generalized version of this problem, that we call k-neighborhood broadcasting, or k-NB (resp. k-neighborhood gossiping, or k-NG) ; here, broadcasting (resp. gossiping) must be realized at distance k in the graph G, $1 \leq k \leq D(G)$, where $D(G)$ denotes the diameter of G. The model we use here is *store-and-forward* (messages go to their destination by transiting in the intermediate nodes ; that is, communications always take place between neighbors), *full-duplex* (when two nodes communicate, they can exchange informations in both directions), *1-port* (at each given time, a node can communicate with at most one of its neighbors) and *unit cost* (a communication between two neighbors takes one time unit, or *round*).

When $k = D(G)$, then the k-NB (resp. k-NG) problem is the usual broadcasting (resp. gossiping) problem, and a wide literature on the subject exists (cf. for instance [8]). We note that k-NB and k-NG have also been studied in the particular case when $k = 1$ in two different families of graphs, namely hypercubes [2,1,5] and star graphs [7].
We denote by $t_{NB}(G, u, k)$ the minimum time to achieve k-NB from node u in the graph G under our model. We call the *k-neighborhood broadcast time* of G, or $t_{NB}(G, k)$, the maximum $t_{NB}(G, u, k)$ over all $u \in V(G)$. Similarly, we denote by $t_{NG}(G, k)$ the minimum time to achieve k-NG in G. Let also $N_k(G, u)$ be the number of vertices which are at distance less than or equal to k from u (u included), and $N_k(G) = \max_{u \in V(G)} N_k(G, u)$. We then have the following property.

Property 1 *For any graph G and for any $1 \leq k \leq D(G)$,*

$$t_{NG}(G, k) \geq t_{NB}(G, k) \geq \max\{k, \lceil \log_2(N_k(G)) \rceil\}$$

Proposition 1 *For any graph G, let u be a vertex of G. For any $1 \leq k \leq D(G)$, let $s_k(u)$ be the number of vertices at distance* exactly k *from u. Then :*

(a) If $s_k(u) \geq 2$, $t_{NG}(G, k) \geq t_{NB}(G, k) \geq k + 1$;
(b) If G is bipartite and $s_k(u) \geq k + 2$, $t_{NG}(G, k) \geq t_{NB}(G, k) \geq k + 2$.

In the following, we denote by $G \Box H$ the cartesian product of graphs G and H.

Proposition 2 *For any graphs G and H, and for any $1 \leq k \leq \min\{D(G), D(H), D(G \Box H)\}$:*

(a) $t_{NB}(k, G \Box H) \leq t_{NB}(k, G) + t_{NB}(k, H)$;
(b) $t_{NG}(k, G \Box H) \leq t_{NG}(k, G) + t_{NG}(k, H)$.

2 Neighborhood Broadcasting [6]

Theorem 1 (k-NB in paths) *Let P_n be the path of order n. For any n and $1 \leq k \leq n - 1$,*
$t_{NB}(P_n, k) = k$ if $k > \lfloor \frac{n-1}{2} \rfloor$, and $t_{NB}(P_n, k) = k + 1$ otherwise.

Here, we determine t_{NB} for the family $T_{\Delta,D}$ of trees with maximum degree Δ and diameter D. We denote by $t_{NB}(T_{\Delta,D}, k)$ the maximum k-NB time over all the trees $T \in T_{\Delta,D}$.

Theorem 2 (*k*-NB in trees)

$$t_{NB}(T_{\Delta,D},k) = \begin{cases} k\cdot(\Delta-1)+1 & \text{if } 1\le k\le \lfloor \frac{D}{2}\rfloor \\ \lfloor \frac{D}{2}\rfloor\cdot(\Delta-2)+k & \text{if } \lfloor\frac{D}{2}\rfloor < k \le D \end{cases}$$

Theorem 3 (*k*-NB in cycles) *Let C_n be the cycle of order n. For any $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $t_{NB}(C_n, k) = k$ if n is even and $k = \frac{n}{2}$, and $t_{NB}(C_n, k) = k+1$ otherwise.*

In the following, $G(n_1, n_2)$ (resp. $TG(n_1, n_2)$) denotes the 2-dimensional grid (resp. torus) with n_1 rows and n_2 columns. Due to symmetries, we will always suppose $n_1 \geq n_2$. We also denote by $D(G(n_1, n_2))$ the diameter of the 2-dimensional grid $G(n_1, n_2)$ (that is $D(G(n_1, n_2)) = n_1 + n_2 - 2$), and by $D(TG(n_1, n_2))$ the diameter of the 2-dimensional torus $TG(n_1, n_2)$, that is $D(TG(n_1, n_2)) = \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor$.
Exact results have been obtained for the k-NB in grids, where $1 \leq k \leq 3$. Due to the lack of space, we do not detail them here. For $k \geq 4$, we have the following proposition.

Proposition 3 (*k*-NB in grids) *For any integers n_1 and n_2 with $n_1 \geq n_2 \geq 2$ and $n_1 + n_2 \geq 6$, let $\mathrm{eq}(n_1, n_2) = 1$ if $n_1 = n_2$ and 0 otherwise. Let also $t_1 = \left\lfloor \frac{2n_2+n_1}{3} \right\rfloor - 1 - \mathrm{eq}(n_1, n_2)$. We have :*

(a) $k+3 \geq t_{NB}(G(n_1,n_2),k) \geq \begin{cases} k+2 & \text{if } 4 \leq k \leq t_1 \\ k+1 & \text{if } t_1+1 \leq k \leq D(G(n_1,n_2))-1 \end{cases}$

(b) $t_{NB}(G(n_1, n_2), k) = k$ *if* $k = D(G(n_1, n_2))$

Proposition 4 (*k*-NB in tori) *For any integers n_1 and n_2 with $n_1 \geq n_2 \geq 3$, we have :*

(a) $t_{NB}(TG(n_1, n_2), 1) = 3$ *;*
(b) $k+1 \leq t_{NB}(TG(n_1, n_2), k) \leq k+3$ *for any* $2 \leq k \leq D(TG(n_1, n_2)) - 1$ *;*
(c) $t_{NB}(TG(n_1,n_2),k) = \begin{cases} k & \text{if } n_1 \text{ and } n_2 \text{ are even} \\ k+1 & \text{otherwise} \end{cases}$
for any $k = D(TG(n_1, n_2))$*.*

However, it is sometimes possible to improve the lower bound in Proposition 4(b).

Proposition 5 *For any integers $n_1 \geq n_2 \geq 4$ where n_1 and n_2 are both even, let $t_2 = \lfloor \frac{2(n_1+n_2-1)}{5} \rfloor$. In that case, we have $t_{NB}(TG(n_1, n_2), k) \geq k+2$ in the following cases :*

(a) $n_1 \geq 4n_2 - 2$ *and* $k \leq 2n_2 - 2$;
(b) $n_1 = 4n_2 - 4$ *and* $k \leq 2n_2 - 3$;
(c) $n_1 \leq 4n_2 - 6$ *and* $k \leq t_2$.

3 Neighborhood Gossiping

Theorem 4 (k-NG in paths) *Let P_n be the path of order n. For any n and $1 \leq k \leq n-1$, $t_{NG}(P_n, k) = k$ if $k = n-1$ is odd, and $t_{NG}(P_n, k) = k+1$ otherwise.*

As in the previous section, $T_{\Delta,D}$ denotes the family of trees with maximum degree Δ and diameter D ; $t_{NG}(T_{\Delta,D}, k)$ denotes the maximum k-NG time over all the trees $T \in T_{\Delta,D}$.

Theorem 5 (k-NG in trees) *For any fixed Δ, D and k :*

(a) $t_{NG}(T_{\Delta,D}, k) = k \cdot (\Delta - 1) + 1$ *if* $1 \leq k \leq \lfloor \frac{D}{2} \rfloor$;
(b) $\lfloor \frac{D}{2} \rfloor (\Delta - 2) + k \leq t_{NG}(T_{\Delta,D}, k) \leq k \cdot (\Delta - 1) + 1$ *otherwise.*

Theorem 6 (k-NG in cycles) *Let C_n be the cycle of order n. For any $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$,*

$$t_{NG}(C_n, k) = \begin{cases} k & \text{if } n \text{ is even and } k = \frac{n}{2} \\ k+1 & \text{if } n \text{ is even and } k \neq \frac{n}{2} \\ k+2 & \text{otherwise} \end{cases}$$

Proposition 6 (k-NG in grids (lower bounds)) *For any integers n_1 and n_2 with $n_1 \geq n_2 \geq 2$, let $eq(n_1, n_2) = 1$ if $n_1 = n_2$ and 0 otherwise. Let also $t_1 = \left\lfloor \frac{2n_2+n_1}{3} \right\rfloor - 1 - eq(n_1, n_2)$. We have :*

$$t_{NG}(G(n_1, n_2), k) \geq \begin{cases} k+2 & \text{if } 4 \leq k \leq t_1 \\ k+1 & \text{if } t_1 + 1 \leq k \leq D(G(n_1, n_2)) - 1 \\ k & \text{if } k = D(G(n_1, n_2)) \end{cases}$$

Since $G(n_1, n_2)$ is the cartesian product of P_{n_1} by P_{n_2}, we can derive upper bounds for the k-NG time in $G(n_1, n_2)$, by combining the results of Theorem 1 and Proposition 2.

Proposition 7 (k-NG in grids (upper bounds)) *For any $n_1 \geq n_2$ and $1 \leq k \leq D(G(n_1, n_2))$:*

$$
t_{NG}(G(n_1,n_2),k) \leq \begin{cases} 2k & \textit{if } n_1 = n_2 = k+1 \textit{ is even} \\ 2k+1 & \textit{if } n_1 = k+1 \textit{ is even and } n_1 \neq n_2 \\ & \textbf{or } \textit{if } n_2 = k+1 \textit{ is even and } n_1 \neq n_2 \\ 2k+2 & \textit{otherwise} \end{cases}
$$

Proposition 8 (k-NG in tori (lower bounds)) *For any even n_1 and even n_2 with $n_1 \geq n_2 \geq 4$, and any $1 \leq k \leq D(TG(n_1,n_2))$, we have :*

$$
t_{NG}(TG(n_1,n_2),k) \geq \begin{cases} k+2 & \textit{if } n_1 \textit{ and } n_2 \textit{ satisfy the conditions of Prop. 5} \\ k+1 & \textit{if } k \neq D(TG(n_1,n_2)) \\ k & \textit{if } k = D(TG(n_1,n_2)) \end{cases}
$$

$TG(n_1,n_2)$ is the cartesian product of C_{n_1} by C_{n_2}, hence we can derive upper bounds for the k-NG time in $TG(n_1,n_2)$, by combining the results of Theorem 6 and Proposition 2. Indeed, we can show an upper bound for $t_{NG}(TG(n_1,n_2),k)$ lying between $2k$ and $2k+4$, depending on the respective values of n_1, n_2 and k. We also have determined a general k-NG algorithm in $TG(n_1,n_2)$ when n_1 and n_2 are both even, that runs in $2k+1$ rounds. Hence the following proposition.

Proposition 9 *For any even n_1 and even n_2 such that $n_1 \geq n_2 \geq 4$, and for any $1 \leq k \leq D(TG(n_1,n_2))$, $t_{NG}(TG(n_1,n_2),k) \leq 2k+1$.*

For small k, it appears that the upper bound of Proposition 9 above is optimal.

Corollary 1 *For any even n_1 and even n_2, $t_{NG}(TG(n_1,n_2),1) = 3$ and $t_{NG}(TG(n_1,n_2),2) = 5$ if $\min\{n_1,n_2\} \geq 6$.*

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