A Survey on Knödel Graphs

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Abstract

Knödel graphs of even order n and degree $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$, $W_{\Delta,n}$, are graphs which have been introduced some 25 years ago as the topology underlying a time optimal algorithm for gossiping among n nodes [26]. However, they have been formally defined only 7 years ago [17]. Since then, they have been widely studied as interconnection networks, mainly because of their good properties in terms of broadcasting and gossiping [3,14]. In particular, Knödel graphs of order 2^k, and of degree k, are among the three most popular families of interconnection networks in the literature, along with the hypercube of dimension k, H_k [28], and with the recursive circulant graph $G(2^k, 4)$ introduced by Park and Chwa in 1994 [32]. Indeed, those three families are commonly presented as good topologies for multicomputer networks, and are comparable since they have the same number of nodes and the same degree.

In this paper, we first survey the different results that exist concerning Knödel graphs, mostly in terms of broadcasting and gossiping. We complete this survey by a study of graph-theoretical properties of the "general" Knödel graph $W_{\Delta,n}$, for any even n and $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$. Finally, we propose a rather complete study of Knödel graphs $W_{k,2^k}$, which allows to compare this topology to the hypercube of dimension k, H_k, and the recursive circulant graph $G(2^k, 4)$. We also provide a study of the different embeddings that can exist between any two of these topologies.

Key words: Knödel graphs, broadcasting, gossiping, interconnection networks, hypercubes, recursive circulant graphs, graph embeddings.

1 Introduction

Knödel graphs have been originally introduced in 1975 [26] ; they were graphs that were underlying Knödel's construction of a time optimal algorithm for gossiping among n vertices, with even n. However, the family of Knödel graphs has been formally defined some 20 years later, by Fraigniaud and Peters [17]. They are regular graphs of even order n and degree $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$, and they are denoted by $W_{\Delta,n}$. Since 1994, there have been several papers dealing with Knödel graphs, especially because some subfamilies of Knödel graphs tend to have good properties in terms of broadcasting and gossiping ; it also appeared that many of the graphs given as examples of Minimum Broadcast (resp. Gossip) Graphs, such as in [25,6,27], were in fact isomorphic to the Knödel graphs [14].

In particular, for any $n = 2^k$, the Knödel graph of order n and degree k, $W_{k,2^k}$, turns out to be Minimum Broadcast (resp. Gossip, Linear Gossip) Graph. In that sense, $W_{k,2^k}$ compares to the hypercube of dimension k, H_k, and the recursive circulant graph $G(2^k, 4)$ introduced in 1994 by Park and Chwa [32]. It actually makes sense to compare those three topologies, since they all have good properties in terms of interconnection networks, and since they also all are of same order 2^k, and regular of same degree k.

In this paper, we propose to survey the main results known about Knödel graphs, and to give a better understanding of their structure. In particular, we focus on three main topics :

- The properties of Knödel graphs in terms of broadcasting and gossiping, which was historically the starting point of the study of these graphs.
- The graph-theoretical properties of the Knödel graph $W_{\Delta,n}$, for any even n and $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$.
- A deeper study of the particular case $n = 2^k$ and $\Delta = k$. Among others, we provide a comparison, in terms of embeddings, between $W_{k,2^k}$, the hypercube of dimension k, H_k, and the recursive circulant graph $G(2^k, 4)$.

In Section 2, we give different definitions and notations necessary to introduce the problem. We then survey in Section 3 the general known results concerning the properties that Knödel graphs satisfy about broadcasting and gossiping. A graph-theoretical study of Knödel graphs $W_{\Delta,n}$ is then undertaken ; more precisely, Section 4 deals with the "general" Knödel graph $W_{\Delta,n}$ with even n and $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$, while Section 5 only focuses on the particular case $n = 2^k$ and $\Delta = k$.

2 Definitions

In this Section, we first give some definitions and notations concerning the graphs we are going to deal with in this paper. The second part of this Section is devoted to broadcasting and gossiping, as well as the different communication models that we will focus on, and the associated graphs of minimum size.

2.1 The Knödel graph $W_{\Delta,n}$

Definition 1 (Knödel graph [17]) *The* Knödel graph *on $n \geq 2$ vertices (n even) and of maximum degree $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$ is denoted $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the pairs (i,j) with i=$1,2$ and $0 \leq j \leq \frac{n}{2} - 1$. For every j, $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1,j)$ and every vertex $(2, j + 2^k - 1 \text{ } mod \text{ } \frac{n}{2})$, for $k = 0, \ldots, \Delta - 1$.*

For $0 \leq k \leq \Delta - 1$, an edge of $W_{\Delta,n}$ which connects a vertex $(1,j)$ to the vertex $(2, j + 2^k - 1 \bmod \frac{n}{2})$ is said to be *in dimension k*. For a better understanding, we give some examples of Knödel graphs in Figure 1.

We note that when $\Delta = 1$, $W_{1,n}$ consists in $\frac{n}{2}$ (disconnected) copies of K_2. Actually, $W_{\Delta,n}$ is connected iff $\Delta \geq 2$, since in that case it suffices to alternate edges in dimension 0 and 1 to get a Hamiltonian cycle.

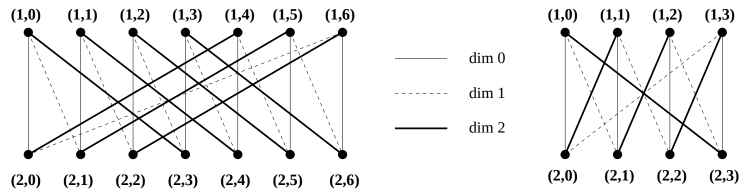

Fig. 1. $W_{14,3}$ and (right) $W_{8,3}$

Now let us define the notion of *bipartite incident graph*, which, we will see, is related to Knödel graphs.

Definition 2 (Bipartite Incident Graph) *Let $G = (V, A)$ be a directed graph of order n, with $V = \{0, 1, \ldots, n-1\}$. The* bipartite incident graph *of G is the bipartite graph $H = (V_1, V_2, E)$ of order $2n$, where $V_i = \{0_i, 1_i, \ldots, (n-1)_i\}$ for any $i \in \{1, 2\}$, and such that for any arc $(x, y) \in A$, there corresponds an edge $x_1 y_2 \in E$, with $x_1 \in V_1$ and $y_2 \in V_2$. Moreover, for all $x \in V$, there is an edge $x_1 x_2 \in E$, where $x_1 \in V_1$ and $x_2 \in V_2$.*

Thanks to the definition above, it is not difficult to see that the Knödel graph $W_{\Delta,n}$ is the bipartite incident graph of a digraph $G = (V, A)$ of order $\frac{n}{2}$. Indeed, let $V = \{j \mid 0 \leq j \leq \frac{n}{2} - 1\}$, and $V_i = \{(i,j) \mid 0 \leq j \leq \frac{n}{2} - 1\}$ for $i \in \{1, 2\}$. In order for G to be such that $W_{\Delta,n}$ is the bipartite incident graph of G, we see that every arc (x, y) in G must satisfy the following relation : $\exists$ $p \in \{1, \ldots, \Delta - 1\}$ *s.t.* $y \equiv x + 2^p - 1 \bmod \frac{n}{2}$.
More precisely, we see that $W_{\Delta,n}$ is the bipartite incident graph of a circulant

digraph G ; we also say that $W_{\Delta,n}$ is the *bi-circulant* graph of G. Below is a definition of a circulant graph and digraph.

Definition 3 (Circulant graph/digraph) *A* circulant graph *(resp.* digraph*) on n vertices $C_n(a_1, a_2, \dots a_p)$ (resp. $\vec{C}_n(a_1, a_2, \dots a_p)$), with $a_i \in \mathbb{N}^*$ and $a_1 < a_2 < \dots < a_p$, has vertex set $V = \{0, 1, \dots, n-1\}$ and edge set (resp. directed edge set) $E = \{xy \mid \exists\ a_i,\ 1 \leq i \leq p$ such that $x + a_i \equiv y\ (mod\ n)\}$.*

Consequently, we see here that $W_{\Delta,n}$ is the *bi-circulant* of $\vec{C}_{\frac{n}{2}}(1, 3, \dots, 2^{\Delta-1} - 1)$. An example of such a relation between $W_{\Delta,n}$ and $\vec{C}_{\frac{n}{2}}(1, 3, \dots, 2^{\Delta-1} - 1)$ is given in Figure 2, where $n = 14$ and $\Delta = 3$.

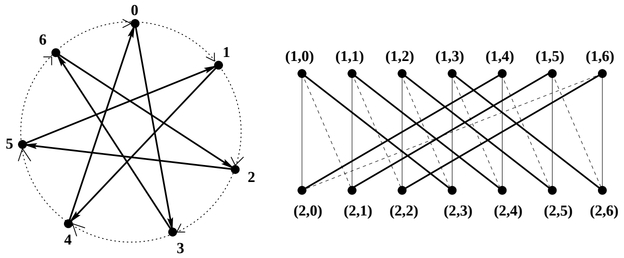

Fig. 2. $\vec{C}_7(1, 3)$ and (right) $W_{3,14}$, its corresponding bi-circulant

Finally, let us give here the definition of the recursive circulant graph $G(n, d)$, which will be useful in order to compare performances of the hypercube H_k, the Knödel graph $W_{k,2^k}$ and $G(2^k, 4)$ in Section 5. An example of a recursive circulant graph $G(2^k, 4)$, with $k = 4$, is given in Figure 3.

Definition 4 (Recursive Circulant Graphs $G(n, d)$ [32]) *The recursive circulant graphs $G(n, d)$ with $d \geq 2$, are defined as follows. The vertex set is $V = \{0, 1, 2, \dots n-1\}$, and the edge set is $E = \{uv \mid \exists\ i,\ 0 \leq i \leq \lceil log_d(n) \rceil - 1$, such that $u + d^i \equiv v\ (mod\ n)\}$.*

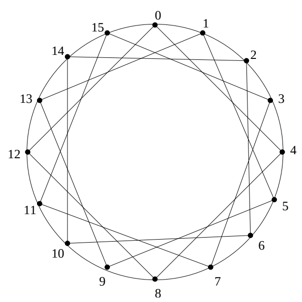

Fig. 3. The recursive circulant graph $G(16, 4)$

2.2 Broadcasting and Gossiping

In this Section, we define the notions of broadcasting and gossiping, as well as the associated models and minimum graphs (for rather complete surveys on the topic, we refer to [22,16,24]). This will allow us to survey in Section 3 the properties of Knödel graphs with respect to these communication models.

Let us consider an interconnection network, where each node knows some piece of information. We will model such a network by a graph, where vertices represent the nodes and edges the communication links. Broadcasting (resp. gossiping) refers to the task whereby one vertex (resp. every vertex) knows a piece of information and needs to transmit it to every other vertex in the network. Here, we will always consider a *1-port* and *store-and-forward* model, that is, respectively :

- one vertex communicates with only one of its neighbours at any given time ;
- messages progress in the network towards their destination, transiting in intermediate vertices. Hence, at each step, the used link is immediately freed.

We will then consider different cases :

- The model can be either *unit cost* or *linear cost*. In the latter, the time to communicate between two vertices u and v includes a fixed start-up time β, and also a propagation time $L\tau$ which is proportional to the length L of the message exchanged ; while in the former, it will always take one time unit to communicate, no matter how long the message is ;
- The model can also be either *simplex* or *full-duplex*. In the latter, when two vertices communicate, the communication can flow in both directions. In that case, the graph modelling the network is *undirected*.
  In the former, a communication link can be used to send messages only in a particular fixed direction ; hence the network is modelled by a *directed graph*.

Note that in this paper, we will not consider the case of gossiping in the *simplex* model.
Whatever the considered model, we have the following definitions. If we consider a graph G :

- For a vertex u of G, the *broadcast time of* u, is the minimum time needed by u to broadcast its information in the graph G. It is denoted by $b(u)$ in the *full-duplex* model, and by $\vec{b}(u)$ in the *simplex* model.
  In the *full-duplex* model, the *broadcast time* of G, or $b(G)$ is defined as follows : $b(G) = \max\{b(u) \mid u \in V(G)\}$. Analogously, in the *simplex* model, the broadcast time of G, or $\vec{b}(G)$, is defined as follows : $\vec{b}(G) = \max\{\vec{b}(u) \mid u \in V(G)\}$.
- The *gossip time of* G defines the minimum time to gossip in G. It is denoted by $g(G)$ in the *unit cost* model, and by $g_{\beta,\tau}(G)$ in the *linear cost* model.

Starting from this point, we will call *broadcast graph* any graph G such that $b(G) = b(K_n)$, where K_n is the complete graph of order n. Analogously :

- a *broadcast digraph* denotes a graph G such that $\vec{b}(G) = \vec{b}(K_n^*)$ (where K_n^* is the complete directed graph of order n) ;
- a *gossip graph* denotes a graph G such that $g(G) = g(K_n)$;
- a *linear gossip graph* denotes a graph G such that $g_{\beta,\tau}(G) = g_{\beta,\tau}(K_n)$.

The following results give the minimum time to communicate in the complete (directed) graph under the four models of communication defined above.

Property 1

- $b(K_n) = \lceil \log_2(n) \rceil$ *[7].*
- $\vec{b}(K_n^*) = \lceil \log_2(n) \rceil$ *[29].*
- $g(K_n) = \lceil \log_2(n) \rceil + odd(n)$*, where* $odd(n) = 1$ *if* n *is odd, and* 0 *otherwise [26].*
- *For any even* n*,* $g_{\beta,\tau}(K_n) = \lceil \log_2(n) \rceil \beta + (n-1)\tau$ *[17].*

Remark 1 *We note that when* n *is odd, the optimal gossip time under the linear cost model* $g_{\beta,\tau}(n)$ *has been determined in [13], but is much more complex. Notably, it depends on the value of* n *in the range* $[2^{k-1}+1; 2^k-1]$.

However, in all these models, it is not always necessary to consider the complete (directed) graph to get a broadcast (gossip (linear)) (di)graph. In that case, we can define the following :

- a *Minimum Broadcast Graph* is a broadcast graph with a minimum number of edges ;
- a *Minimum Broadcast Digraph* is a broadcast digraph with a minimum number of arcs ;
- a *Minimum Gossip Graph* is a gossip graph with a minimum number of edges ;
- a *Minimum Linear Gossip Graph* is a linear gossip graph with a minimum number of edges.

3 Broadcasting and Gossiping in Knödel Graphs

Thanks to the definitions and notations given in the previous Section, we can now survey the results obtained concerning Minimum Broadcast (Di)Graphs and Minimum (Linear) Gossip Graphs in different models, with respect to Knödel graphs. These results are summarized in Tables 1 and 2.

It appears that Knödel graphs are omnipresent in the models presented here. Moreover, it has been shown that some of the graphs that were given in the literature as examples of Minimum Broadcast (resp. Gossip) Graphs (cf. for instance [25,6,27]), are in fact isomorphic to the Knödel graphs [14].

Undirected case	
$W_{k,2^k}$	Minimum Broadcast Graph [8] Minimum Gossip Graph [26] Minimum Linear Gossip Graph [17]
$W_{k-1,2^k-2}$	Minimum Broadcast Graph [6,25,14] Minimum Gossip Graph [27,14] Minimum Linear Gossip Graph [17]
$W_{k-1,2^k-4}$	Minimum Gossip Graph [27,14] Minimum Linear Gossip Graph [17]
$W_{k-1,2^k-6}$	Minimum Linear Gossip Graph [17]
$W_{k-2,n}$ $2^{k-1}+2 \leq n \leq 3 \cdot 2^{k-2}-4$	Broadcast Graph [11] Linear Gossip Graph [12] Gossip Graph [11]
$W_{k-1,n}$ $3 \cdot 2^{k-2}-4 \leq n \leq 2^k-2$	Broadcast Graph [11] Linear Gossip Graph [12] Gossip Graph [11]

Table 1
Broadcasting and gossiping properties of Knödel graphs in the undirected case

In order to understand fully Table 2, which deals with the directed case, we need to introduce the following notion.

Definition 5 *Let G be an undirected graph. We will call the* directed graph *G^* (associated to G), the graph G to which every (undirected) edge uv has been replaced by a pair of symmetric directed edges (u,v) and (v,u).*

In Section 2.1, we have seen that $W_{\Delta,n}$ is a *bi-circulant* graph. Moreover, for $n=2^k$, $n=2^k-1$ and $n=2^k-2$, circulant digraphs $\vec{C}_n(1,3,\ldots,2^{\lfloor\log_2(n)\rfloor}-1)$ have been shown to be Minimum Broadcast Digraphs [10]. We note that if we take the *bi-circulant* of these graphs, we then get the Knödel graphs $W_{\lfloor\log_2(2n)\rfloor,2n}$, which are themselves Minimum Gossip Graphs and Minimum Linear Gossip Graphs. For the first two cases, we also know that they are Minimum Broadcast Graphs (and Digraphs when the edges are replaced by arcs in both directions).

Directed case	
$W^*_{k,2^k}$	Minimum Broadcast Digraph [10]
$W^*_{k-1,2^k-2}$	Minimum Broadcast Digraph [10]
$W^*_{k-2,n}$ $2^{k-1}+2 \leq n \leq 3 \cdot 2^{k-2}-4$	Broadcast Digraph
$W^*_{k-1,n}$ $3 \cdot 2^{k-2}-4 \leq n \leq 2^k-2$	Broadcast Digraph

Table 2
Broadcasting and gossiping properties of Knödel graphs in the directed case

Moreover, Bermond et al. [3] have studied Knödel graphs and their routing, broadcasting and gossiping performances using at each step only edges in a certain dimension i. More precisely, they have studied what they called *Modified Knödel Graphs*, which turn out to be isomorphic to Knödel graphs $W_{\lfloor \log_2(n) \rfloor,n}$ according to Definition 1, for any n not a power of 2.
Their main goal was to study the performances of these graphs, when dimensions are used alternatively. What they proved is that, in some sense, the dimensions of Knödel graphs had a similar role than the ones of hypercubes, with respect to routing, broadcasting and gossiping.

All these results have given us motivations to go further in the study of Knödel graphs, and to compare them with other topologies such as the hypercube. This study is undertaken in the following two sections.

4 Study of Knödel graphs $W_{\Delta,n}$

First, it is easy to see that, for any even n and $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$, $W_{\Delta,n}$ is Δ-regular and bipartite. This is always true by definition. Moreover, any $W_{\Delta,n}$ can be defined as a Cayley graph [23], as recalled in Proposition 1 below.

Proposition 1 ([23]) *For any even n and $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$, $W_{\Delta,n}$ is a Cayley graph on the semi-direct product $G = \mathbb{Z}_2 \ltimes \mathbb{Z}_{\frac{n}{2}}$ for the multiplicative law : $(x,y)(x',y') = (x+x', y+(-1)^x y')$, with $x,x' \in \mathbb{Z}_2$ and $y,y' \in \mathbb{Z}_{\frac{n}{2}}$, and with the set of generators $S = \{(1, 2^i - 1), 0 \leq i \leq \Delta - 1\}$.*

Corollary 1 *For any even n and $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$, $W_{\Delta,n}$ is vertex-transitive.*

Proof : This follows directly from Proposition 1 above, since it is well-known that any Cayley graph is vertex-transitive (see for instance [5]). □

Proposition 2 *For any even n and $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$, the Knödel graph $W_{\Delta,n}$:*

(a) Has edge-connectivity $\lambda(W_{\Delta,n}) = \Delta$;
(b) Has vertex-connectivity $\frac{2\Delta}{3} < \kappa(W_{\Delta,n}) \leq \Delta$.

Proof : (a). It is well known that any vertex-transitive graph G satisfies $\lambda(G) = d(G)$, where $d(G)$ is the degree of G (cf. for instance [5]). Since we know that $W_{\Delta,n}$ is vertex-transitive by Corollary 1, we have $\lambda(W_{\Delta,n}) = \Delta$.
(b). It is also known that, for any graph G, $\kappa(G) \leq \lambda(G)$. Hence we have $\kappa(W_{\Delta,n}) \leq \Delta$. Moreover, Watkins [35] has shown that in any vertex-transitive graph G, regular of degree r, we have $\kappa(G) > \frac{2r}{3}$. Since $W_{\Delta,n}$ is vertex-transitive and regular of degree Δ, we have $\kappa(W_{\Delta,n}) > \frac{2\Delta}{3}$. □

Now we show how to construct $W_{\Delta+1,2m}$ from two copies of $W_{\Delta,m}$, for any even m and $1 \leq \Delta \leq \lfloor \log_2(m) \rfloor$. This will imply several interesting properties, as we will notably see in Section 5.

Proposition 3 *For any even m and $1 \leq \Delta \leq \lfloor \log_2(m) \rfloor$, it is possible to construct $W_{\Delta+1,2m}$ by taking two copies of $W_{\Delta,m}$ and linking the vertices of each copy by a certain perfect matching.*

We note that, in particular, if $2m = 2^k$ and $\Delta + 1 = k$, this gives us a recursive construction of $W_{k,2^k}$, which starts from K_2.

Proof : Let us take two copies W_1 and W_2 of $W_{\Delta,m}$. We will use a bijection f from the set of vertices of W_i, $i \in \{1,2\}$, to the set of vertices of $W_{\Delta+1,2m}$ (we refer to Figure 4 for a better understanding of the process). f is defined as follows on the vertices of W_1 :

- $f((1,i)) = (1,2i)$ for every $i \in [0; \frac{m}{2} - 1]$;
- $f((2,i)) = (2,2i+1)$ for every $i \in [0; \frac{m}{2} - 1]$.

f is defined as follows on the vertices of W_2 :

- $f((1,i)) = (1,2i+1)$ for every $i \in [0; \frac{m}{2} - 1]$;
- $f((2,i)) = (2,2i+2)$ for every $i \in [0; \frac{m}{2} - 1]$.

Now let PM be the following perfect matching between the vertices of W_1 and W_2 : PM consists in adding edges $f((1,i))\ f((2,i))$ for every $i \in [0; m-1]$.
In that case, take a copy of W_1 and a copy of W_2, apply f on the vertices of W_1 and W_2, and add the perfect matching PM : this gives a new graph that is isomorphic to $W_{\Delta+1,2m}$. Indeed, applying f on the vertices of W_1 and W_2 corresponds in reality to a shift in the dimensions : dimension i in $W_{\Delta,m}$ will

become dimension $i+1$ in $W_{\Delta+1,2m}$ for every $i \in [0; \Delta - 1]$. Indeed, for any two adjacent vertices of W_1, $(1, i)$ and $(2, j)$, there must exist a $r \in [0; \Delta - 1]$ such that $j = i + 2^r - 1 \bmod \frac{m}{2}$. By f, we then must have $2j + 1 = 2(i + 2^r - 1) + 1 \bmod m$, i.e. $2j + 1 = 2i + 2^{r+1} - 1 \bmod m$. It is easy to see that the same goes for W_2 : dimension i in W_2 will become dimension $i + 1$ in $W_{\Delta+1,2m}$.
Hence it suffices to add dimension 0 (i.e. the perfect matching PM) to get $W_{\Delta+1,2m}$. □

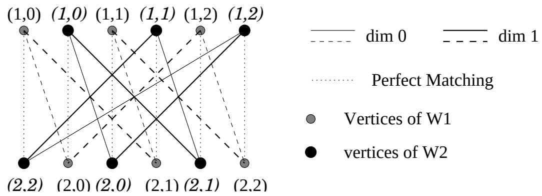

Fig. 4. Constructing $W_{\Delta+1,2m}$ from $W_{\Delta,m}$

Finally, we note the following Proposition, due to [4].

Proposition 4 ([4]) *For all even n, there exists an algorithm to recognize the Knödel graph* $W_{\lceil \log_2(n) \rceil, n}$*, with complexity :*

- $O(n \log_2^3(n))$ *if* $n = 2^k$ *;*
- $O(n \log_2^5(n))$ *otherwise.*

5 Study of $W_{k,2^k}$

5.1 General Properties of $W_{k,2^k}$

In this Section, we focus on the family of Knödel graphs $W_{k,2^k}$. Among others, we will compare this family to the recursive circulant graph $G(2^k, 4)$ and to the hypercube H_k, mainly in terms of performances as interconnection networks. We refer to Figure 5 of Section 5.4 for a comparison between those three families of graphs.

The problem of determining the diameter D_k of $W_{k,2^k}$ has been undertaken in [15]. The result, obtained by a non trivial proof using decomposition of integers into sums of (positive or negative) powers of 2, is the following.

Theorem 1 ([15]) *Let* D_k *be the diameter of* $W_{k,2^k}$*. For any* $k \geq 2$, $D_k = \lceil \frac{k+2}{2} \rceil$.

Now let us check that $W_{k,2^k}$ is non isomorphic to the two other topologies.

Proposition 5 **([14])**

(a) The Knödel graph $W_{k,2^k}$ and the hypercube H_k are non-isomorphic graphs for any $k \geq 4$.
(b) The Knödel graph $W_{k,2^k}$ and the recursive circulant graph $G(2^k, 4)$ are non-isomorphic graphs for any $k \geq 3$.

Proof : Note that for any $k \leq 3$, it can be checked that $W_{k,2^k}$ and H_k are isomorphic. Similarly, for any $k \leq 2$, $W_{k,2^k}$ and $G(2^k, 4)$ are isomorphic.
(a). In order to prove the first part of the Proposition, we use Theorem 1, and note that $k > \lceil \frac{k+2}{2} \rceil$ for any $k \geq 4$. Since the diameter of H_k is equal to k for any $k \geq 1$, we conclude that the diameters of H_k and $W_{k,2^k}$ differ for any $k \geq 4$, and thus those two graphs cannot be isomorphic.
(b). For the second part of the Proposition, it is not sufficient to use the diameter argument, because diameter of $G(2^k, 4)$ is equal to $\lceil \frac{3k-1}{4} \rceil$, and for instance this value coincides with $\lceil \frac{k+2}{2} \rceil$ in the case $k = 4$. However, we can easily see that for any $k \geq 3$, $W_{k,2^k}$ is a bipartite graph, while $G(2^k, 4)$ is not. Indeed, for any $k \geq 3$, there exists in $G(2^k, 4)$ a cycle of length 5, namely $0 - 1 - 2 - 3 - 4 - 0$. □

Proposition 6 *The Knödel graph $W_{k,2^k}$ is not edge-transitive for any $k \geq 4$.*

Proof : First, note that $W_{k,2^k}$ is edge-transitive for any $k \leq 3$, since in that case $W_{k,2^k}$ is isomorphic to the hypercube H_k.
Now suppose $k \geq 4$. The key idea here is the following : in $W_{k,2^k}$, let us consider an edge uv and let us compute N_{uv}, the number of distinct cycles of length 4 that contain uv. If, for two edges uv and $u'v'$, this number differs, then the graph $W_{k,2^k}$ cannot be edge-transitive. Now let us consider uv in dimension 0, and $u'v'$ in dimension 1, and let us show that $N_{uv} \neq N_{u'v'}$.
Let us first consider uv in dimension 0. W.l.o.g., let $u = (1, 0)$ and $v = (2, 0)$. Suppose this edge lies in a cycle of length 4, C_4. Hence, this cycle is of the form $(1, 0) - (2, p) - (1, q) - (2, 0) - (1, 0)$, with $p \neq 0$ and $q \neq 0$. By definition of the Knödel graph $W_{k,2^k}$, this means :

- $p = 2^i - 1 \bmod \frac{n}{2}$ for some $i \neq 0$;
- $q = p - 2^j + 1 \bmod \frac{n}{2}$ for some $j \neq i$;
- $0 = q + 2^l - 1 \bmod \frac{n}{2}$ for some $l \neq 0$.

Altogether, this gives $2^i - 2^j + 2^l - 1 = 0 \bmod \frac{n}{2}$. Since $i \neq 0$ and $l \neq 0$, this implies that $2^j - 1$ is even, hence $j = 0$. Hence we have $2^i + 2^l - 2 = 0 \bmod \frac{n}{2}$. There are only two solutions to this equation : $i = 1$ and $l = k - 1$, or $i = k - 1$ and $l = 1$. These solutions are distinct, since $k - 1 \neq 1$ by hypothesis. Consequently, $N_{uv} = 2$.

Now let us consider an edge $u'v'$ in dimension 1. W.l.o.g., let $u' = (1, 0)$ and $v' = (2, 1)$. If we are looking for a C_4 of the form $(1, 0) - (2, p) - (1, q) -$

$(2,1)-(1,0)$ (with $p \neq 1$ and $q \neq 0$), we get the following equations :

- $p = 2^i - 1 \bmod \frac{n}{2}$ for some $i \neq 1$;
- $q = p - 2^j + 1 \bmod \frac{n}{2}$ for some $j \neq i$;
- $1 = q + 2^l - 1 \bmod \frac{n}{2}$ for some $l \neq 1$.

In that case, we can see that it is possible to find at least three distinct solutions for the triplet (i,j,l). They are the following : $\{(0,k-1,0),(2,1,k-1),(k-1,1,2)\}$. Since $k \geq 4$, these 3 solutions are pairwise distinct. This shows that $N_{u'v'} \geq 3$, and consequently $W_{k,2^k}$ is not edge transitive for any $k \geq 4$. □

Remark 2 *It has been shown independently in [4] that for any even n such that $n \neq 2^k - 2$, $W_{k-1,n}$ is not edge-transitive, while in the case $n = 2^k - 2$, $W_{k-1,n}$ is edge-transitive [23].*
Moreover, when $n = 2^k - 2$, it is possible to show that the vertex-connectivity of $W_{k-1,n}$ is maximum, that is $\kappa(W_{k-1,2^k-2}) = k-1$ [23].

5.2 Forwarding Indices, Bisection Width and Optical Index

Definition 6 (Compound Graph [5]) *A* compound graph *of a graph G by a graph H is a graph obtained the following way : we replace the vertices of G by copies of H, and we add edges to some of the vertices of two of these copies iff the corresponding vertices of G are adjacent.*

In the following, we will denote by $G[H]$ any compound graph of a graph G by a graph H, where there is exactly one perfect matching between two copies of H iff the corresponding vertices of G are adjacent. This operation clearly does not define a unique graph ; however, in the following we will write $G' = G_1[G_2]$ if G' is a compound graph of the form $G_1[G_2]$.

Remark 3

(a) $W_{\Delta+1,2m} = K_2[W_{\Delta,m}]$ *for any even m and $1 \leq \Delta \leq \lfloor \log_2(m) \rfloor$;*
(b) In particular, $W_{k,2^k} = K_2[W_{k-1,2^{k-1}}]$ *for any $k \geq 2$; by induction, this gives* $W_{k,2^k} = K_2[K_2[K_2[\ldots]]]$ *(k times).*

These two statements are direct consequences of Proposition 3.

The vertex-forwarding index, edge-forwarding index and bisection width are useful parameters to judge the routing performances of an interconnection network. Those three notions are defined below.
The notion of forwarding index has been introduced in [18]. A *routing* R of a graph G of order n is a set of $n(n-1)$ elementary paths $R(u,v)$ specified for all ordered pairs (u,v) of vertices of G. If all the paths of $R(u,v)$ are shortest

paths from u to v, then the routing is said to be a *routing of shortest paths*. Let the *load of a vertex* v in a given routing R of a graph G, denoted by $\xi(G, R, v)$, be the number of paths of R going through v (where v is not an end vertex). A routing for which the load of all vertices is the same is called a *vertex-uniform routing*. The *vertex-forwarding index of a network* (G, R), denoted by $\xi(G, R)$, is the maximum number of paths of R going through any vertex v in $V(G) : \xi(G, R) = \max_{v \in V(G)} \xi(G, R, v)$. The minimum vertex-forwarding index over all possible routings of a graph G is denoted $\xi(G)$ and is called the *vertex-forwarding index* of $G : \xi(G) = \min_R \xi(G, R)$.
Similar definitions hold for the edge-forwarding index of a graph G (cf. [30]) : the *load of an edge* e in a given routing R of a graph G, denoted by $\Pi(G, R, e)$, is the number of paths of R going through e. A routing for which the load of all edges is the same is called an *edge-uniform routing*. The *edge-forwarding index of a network* (G, R), denoted by $\Pi(G, R)$, is the maximum number of paths of R going through any edge e in $E(G) : \Pi(G, R) = \max_{e \in E(G)} \Pi(G, R, e)$. The minimum edge-forwarding index over all possible routings of a graph G is denoted $\Pi(G)$ and is called the *edge-forwarding index* of $G : \Pi(G) = \min_R \Pi(G, R)$.
The *edge-bisection width*, $Bisw(G)$, of a graph G of order N is defined as the minimum number of edges whose removal splits the graph into two subgraphs holding roughly half the number N of vertices. Formally, $Bisw(G)$ is the minimum number of edges running between V_1 and V_2, over all partitions of the vertex set of G into two parts $V_1 \cup V_2$ such that $|V_1| = |V_2| \pm 1$.

Proposition 7

(a) *The edge-forwarding index of* $W_{k,2^k}$, $\Pi(W_{k,2^k})$, *satisfies* $\Pi(W_{k,2^k}) = 2^k$ *for any* k.
(b) *The vertex-forwarding index of* $W_{k,2^k}$, $\xi(W_{k,2^k})$, *satisfies* $\xi(W_{k,2^k}) = \frac{1}{2^k} \sum_{(u,v) \in V \times V} d(u, v) - (2^k - 1)$ *for any* k.
(c) *The edge-bisection width of* $W_{k,2^k}$, $Bisw(W_{k,2^k})$, *satisfies* $Bisw(W_{k,2^k}) = 2^{k-1}$ *for any* k.

Proof : (a). In [20], Gauyacq proved that if G' is the compound graph of H by G such that for every edge uv of H there is at least one perfect matching between the corresponding copies G_u and G_v, then we have :

$$\frac{n_G^2 \Sigma(H)}{e_{G'}^{(H)}} \leq \Pi(G') \leq max\{n_G \Pi(H), n_H \Pi(G)\} \textbf{ (I1)}.$$

where n_G is the order of the graph G, $e_{G'}^{(H)}$ is the number of edges of type H in the graph G', and where $\Sigma(H) = \sum_{(u,v) \in V(H) \times V(H)} d_H(u, v)$.
In our case, we have seen that $W_{k,2^k} = K_2[W_{k-1,2^{k-1}}]$ for any $k \geq 1$. Since K_2 has only one edge, it is not difficult to see that there is a perfect matching between the two copies of $W_{k-1,2^{k-1}}$. Hence we can apply inequality **(I1)**, where $H = K_2$, $G = W_{k-1,2^{k-1}}$ and $G' = W_{k,2^k}$. Note first that :

- $\Sigma(K_2) = 2$;
- $e_{G'}^{(K_2)} = 2^{k-1}$;
- $\Pi(K_2) = 2$.

In that case, **I1** becomes : $2^k \leq \Pi(W_{k,2^k}) \leq \max\{2^k, 2 \cdot \Pi(W_{k-1,2^{k-1}})\}$ for any $k \geq 1$ **(I2)**. Now it is easy to prove by induction that $\Pi(W_{k,2^k}) = 2^k$ for any $k \geq 1$. Indeed, $\Pi(W_{k,2^k}) = 2^k$ for any $k \leq 3$, since $W_{k,2^k}$ is isomorphic to the hypercube H_k for any $k \leq 3$, and since $\Pi(H_k) = 2^k$ for any k [5]. Moreover, suppose we have $\Pi(W_{p,2^p}) = 2^p$ for some p. By **(I2)**, we then have : $2^{p+1} \leq \Pi(W_{p+1,2^{p+1}}) \leq \max\{2^{p+1}, 2 \cdot 2^p\}$, hence the result. Note that this value is the same as for the hypercube H_k.

(b). Concerning the vertex-forwarding index, we have the following from [5]. Since $W_{k,2^k}$ is a Cayley graph, there exists a vertex-uniform shortest-paths routing, and consequently we have : $\xi(W_{k,2^k}) = \frac{1}{2^k} \sum_{(u,v) \in V \times V} d(u,v) - (2^k - 1)$. We refer to Table 3 for the comparison of $\xi(H_k)$ and $\xi(W_{k,2^k})$ for $2 \leq k \leq 6$.

k	$\xi(H_k)$	$\xi(W_{k,2^k})$
2	1	1
3	5	5
4	17	15
5	49	41
6	129	103

Table 3
Comparison between $\xi(H_k)$ and $\xi(W_{k,2^k})$ for $2 \leq k \leq 6$

(c). Finally, let us prove that the edge-bisection width satisfies $Bisw(W_{k,2^k}) = 2^{k-1}$. First, we have the following inequality from [5] : $\Pi(W_{k,2^k}) \cdot Bisw(W_{k,2^k}) \geq \frac{(2^k)^2}{2}$. Since we know $\Pi(W_{k,2^k}) = 2^k$, we have $Bisw(W_{k,2^k}) \geq 2^{k-1}$. Moreover, it is easy to see that $Bisw(W_{k,2^k}) \leq 2^{k-1}$, since by Proposition 3 we know that $W_{k,2^k}$ can be constructed with two (disconnected) copies of $W_{k-1,2^{k-1}}$ joined by a perfect matching PM. Since this perfect matching has 2^{k-1} edges, we have the equality. □

The minimum number of wavelengths necessary to achieve all-to-all communications in a graph G is an essential parameter for optical networks [34]. A good survey on the problems related to optical networks can be found in [2]. Consider the directed symmetric Knödel graph $W^*_{k,2^k}$ associated to $W_{k,2^k}$. Suppose that we want to join each pair of vertices in $W^*_{k,2^k}$ by a dipath, and that we construct a routing which satisfies this request. Moreover, we suppose that we give a color to every dipath of the routing, where several dipaths can have the

same color. The main question here is to determine the *optical index* $w(W_{k,2^k})$, that is the minimum number of colors, over all the possible routings, which are necessary to color all the dipaths, and such that two dipaths sharing the same arc have different colors.
This corresponds to the number of wavelengths which are necessary for all-to-all communications in an optical network using the WDM (Wavelength Division Multiplexing) technique.

Proposition 8 *The optical index of* $W_{k,2^k}$*,* $w(W_{k,2^k})$*, satisfies* $w(W_{k,2^k}) = 2^{k-1}$ *for all* $k \geq 2$.

Proof : In [1], Amar et al. proved the following. Let p_i be integers for $i \in [1;n]$ such that $2 \leq p_1 \leq p_2 \leq \ldots \leq p_n$, then we have $w(K_{p_1}[K_{p_2}[\ldots[K_{p_n}]]\ldots]) = \prod_{i=2}^{n} p_i$. In our case, we know that $W_{k,2^k} = K_2[K_2 \ldots [K_2]] \ldots]$ (k times) by Remark 3. Hence each $p_i = 2$ for $i \in [1;n]$, and we have $w(W_{k,2^k}) = 2^{k-1}$. □

5.3 Bipancyclicity

Definition 7 (Bipancyclicity) *A graph of order n is said to be* bipancyclic *when it holds cycles of any even length* $4 \leq 2m \leq n$.

Proposition 9 $W_{k,2^k}$ *is bipancyclic for any* $k \geq 2$.

Proof : Note that since $W_{k,2^k}$ is bipartite, it cannot hold any cycle of odd length. Now let us show that for any even number $4 \leq 2m \leq n$, it is possible to find a cycle of length $2m$. The key idea here is to use only three different dimensions to get those cycles, namely dimensions 0, 1 and $k-1$. Now let us detail two cases, depending on the parity of m :

- If m is odd, it is possible to get a cycle of order $2m$, which is the following : $(1,0)-(2,0)-(1,1)-(2,2)-(1,3)-(2,4)-\ldots-(1,2p+1)-(2,2p+2)-\ldots-(1,m-1)-(2,m-1)-(1,m-2)-(2,m-3)-\ldots-(1,2p)-(2,2p-1)-\ldots-(2,1)-(1,0)$. In other words, we use the following sequence of dimensions :

  dim 0-[dim $(k-1)$-dim 1]$^{\frac{m-1}{2}}$-dim 0-[dim 1-dim $(k-1)$]$^{\frac{m-1}{2}}$.

  where [dim a-dim b]^c means that we alternate edges in dimension a and b, c times.
- If m is even, it is possible to get a cycle of order $2m$, which is the following : $(1,0)-(2,0)-(1,1)-(2,2)-(1,3)-(2,4)-\ldots-(1,2p+1)-(2,2p+2)-\ldots-(2,m-1)-(1,m-1)-(2,m-2)-(1,m-3)-\ldots-(2,2p+1)-(1,2p)-\ldots-(2,1)-(1,0)$. In other words, we use the following sequence of dimensions :

  dim 0-[dim $(k-1)$-dim 1]$^{\frac{m}{2}-1}$-dim $(k-1)$-dim 0-[dim 1-dim $(k-1)$]$^{\frac{m}{2}-1}$-dim 1.

For a better understanding of the method, we refer to Figure 5. □

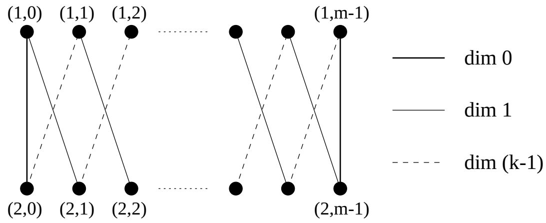

Fig. 5. How to find a cycle of length $2m$ in $W_{k,2^k}$

Definition 8 *A graph G, regular of degree d, is said to be* Hamiltonian decomposable *if it can be decomposed in $\lfloor \frac{d}{2} \rfloor$ Hamiltonian cycles, plus a perfect matching if d is odd.*

Open Problem 1 *Is $W_{k,2^k}$ Hamiltonian decomposable for any k ?*

Remark 4 *We note that the answer is positive for any $k \leq 5$. Indeed, if $k \leq 3$, $W_{k,2^k}$ is isomorphic to the hypercube H_k, and we know that H_k can be decomposed in Hamiltonian cycles for any k. Moreover, for $k = 4$, we have shown the decomposition of Figure 6. This decomposition added to the recursive construction of $W_{5,32}$ from two copies of $W_{4,16}$ (cf. Proposition 3) also gives a Hamiltonian decomposition for $k = 5$.*

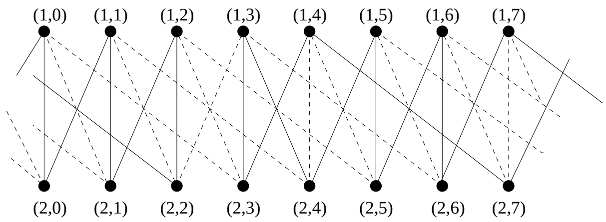

Fig. 6. Decomposition of $W_{4,16}$ into two Hamiltonian cycles

Moreover, coming back to the case where $n \neq 2^k$, we note that we have the following Proposition concerning the Hamiltonian decomposition of $W_{k-1,2^k-2}$.

Proposition 10 *$W_{k-1,2^k-2}$ is Hamiltonian decomposable for any $k \geq 3$.*

Proof : We know by [23] that in the case $n = 2^k - 2$ and $\Delta = k - 1$, $W_{\Delta,n}$ is edge-transitive. In that case, since we know that edges in dimensions 0 and 1 form a Hamiltonian cycle, then any edges in dimensions i and $i+1$ (with $0 \leq i \leq k-3$) also form a Hamiltonian cycle. If we repeat the process for any even $0 \leq i \leq k-3$, then we end up with two cases :

- If $k-1 = 2p$ is even, we get p Hamiltonian cycles and we have used all the edges.
- If $k-1 = 2p+1$ is odd, then the edges in dimension $k-2$ have not been used ; however, they form a perfect matching, and thus $W_{k-1,2^k-2}$ is also Hamiltonian decomposable.

□

5.4 Comparison between $W_{k,2^k}$, H_k and $G(2^k, 4)$

We know by Theorem 1 (cf.[15]) that the Knödel graph $W_{k,2^k}$ has a diameter smaller (by a constant factor) than the one of the hypercube H_k, or of the recursive circulant graph $G(2^k, 4)$ (which was introduced by Park and Chwa [32] as a topology which could compete with the hypercube). The diameter is one of the parameters for which we can say that $W_{k,2^k}$ can compete with H_k and $G(2^k, 4)$. In this Section, we continue the comparison between the three topologies by studying some embeddings from one architecture into another. To these three architectures, we have added a fourth one, namely the recursive circulant graph $G(2^k, 2)$, in order to complete a study that was initiated in [33].
All these results are summarized in Table 4. Finally, we provide in Table 5, a comparison between the three families of graphs.

We note that the embeddings below always take place between graphs having the same number of vertices (that is, $n = 2^k$). Hence, the expansion is always equal to 1.

Proposition 11

(a) For all $k \geq 3$, it is possible to embed $W_{k,2^k}$ into the recursive circulant graph $G(2^k, 2)$ with dilation 2.
(b) For all $k \geq 3$, it is possible to embed the recursive circulant graph $G(2^k, 2)$ into $W_{k,2^k}$ with dilation 2.

Proof : (a). Let us first embed $W_{k,2^k}$ into $G(2^k, 2)$, and let $n = 2^k$. For this, we define the bijection f from $V(W_{k,2^k})$ to $V(G(2^k, 2))$ as follows :

- $f((1, i)) = 2i$ for all $i \in [0; \frac{n}{2} - 1]$;
- $f((2, i)) = 2i + 1$ for all $i \in [0; \frac{n}{2} - 1]$.

An example of such an embedding, with $k = 3$, is illustrated in Figure 7.

Now, we show that this embedding has dilation 2. For this, for any vertex u in $W_{k,2^k}$, and for all the neighbours v of u, we have to show that $d(f(u), f(v)) \leq 2$ in $G(2^k, 2)$. Let us distinguish two cases :

- u is of the form $(1, i)$. Then the neighbours v of u in $W_{k,2^k}$ are the $(2, i+2^l-1 \mod \frac{n}{2})$, with $0 \leq l \leq k-1$. In that case, $f(u) = 2i$. If $l = 0$, $v = (2, i)$ and $f(v) = 2i+1$. Thus, $f(u)$ and $f(v)$ are neighbours in $G(2^k, 2)$. If $l \neq 0$, then $f(v) = 2i + 2^{l+1} - 2 \mod n$. But $2i$ is neighbour of $2i + 2$ in $G(2^k, 2)$, and

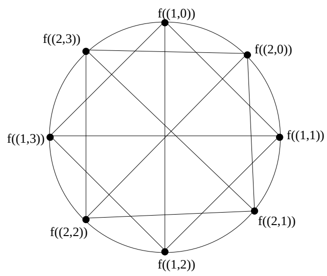

Fig. 7. Embedding of $W_{3,8}$ into $G(8,2)$

$2i+2$ is neighbour of $2i-2^{l+1}+2 \bmod n$. Hence the dilation is equal to 2 in that case.

- If u is of the form $(2,i)$, the neighbours v of u in $W_{k,2^k}$ are of the form $(1, i-2^l+1 \bmod \frac{n}{2})$ with $0 \le l \le k-1$. It is easy to see that, as in the previous case, we have $d(f(u), f(v)) \le 2$ in $G(2^k, 2)$, for all the vertices v described as above.

(b). Now let us realize the opposite embedding, that is from $G(2^k, 2)$ into $W_{k,2^k}$. Let f be the bijection from $V(G(2^k,2))$ to $V(W_{k,2^k})$, defined as follows :

- $f(2i) = (1,i)$ for all $i \in [0; \frac{n}{2}-1]$;
- $f(2i+1) = (2,i)$ for all $i \in [0; \frac{n}{2}-1]$;

An example of such an embedding, with $k=3$, is illustrated in Figure 8.

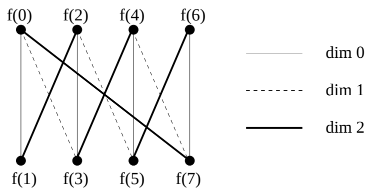

Fig. 8. Embedding of $G(8,2)$ into $W_{3,8}$

In order to show that the dilation of such an embedding is equal to 2 as well, we show that for any vertex u in $G(2^k,2)$ and for all the neighbours v of u, we have $d(f(u), f(v)) \le 2$ in $W_{k,2^k}$. As previously, we consider two cases :

- u is even ($u = 2m$). Then $f(u) = (1,m)$. Moreover, the neighbours v of u in $G(2^k,2)$ are of the form $v = 2m \pm 2^l \bmod n$ with $0 \le l \le k-1$. We distinguish two cases : first, if $l = 0$, then v is odd (either $v = 2m+1$ or $v = 2m-1 = 2(m-1)+1$). Then $f(v) = (2,m)$ or $f(v) = (2, m-1)$ depending on the cases. But we know that $f(u)$ and $f(v)$ are neighbours in $W_{k,2^k}$. Now, if $l \neq 0$, then $v = 2m \pm 2^l \bmod n$ is even, and $f(v) = (1, m \pm 2^{l-1} \bmod \frac{n}{2})$. In that case, $f(u)$ and $f(v)$ are at distance 2, since $(1,m)$ and $(2,m-1)$ (resp. $(1,m)$ and $(2, m+2^l-1 \bmod \frac{n}{2})$) are neighbours in $W_{k,2^k}$.
- u is odd ($u = 2m+1$). In that case, we proceed as previously, and we can show easily that the dilation is also equal to 2.

The embedding of $G(2^k, 2)$ into $W_{k,2^k}$ can then be realized with dilation 2. We note that in this case, the dilation is optimal, since $|E(G(2^k, 2))| > |E(W_{k,2^k})|$ for all $k \geq 2$. □

Proposition 12 *For all $k \geq 4$, there exists an embedding of the Knödel graph $W_{k,2^k}$ into the hypercube of dimension k, H_k, with dilation not exceeding 4.*

Proof : For this, it suffices to combine the result of Proposition 11(a) with the following result, coming from [33] : it is possible to embed $G(2^k, 2)$ into H_k with dilation 2. In that case, if we first embed $W_{k,2^k}$ into $G(2^k, 2)$, and then $G(2^k, 2)$ into H_k, we get an embedding of $W_{k,2^k}$ into H_k with dilation less than or equal to $2 \cdot 2 = 4$. □

Proposition 13 *For all $k \geq 4$, it is possible to embed the hypercube of dimension k, H_k, into $W_{k,2^k}$, with dilation 2.*

Proof : First, we note that $W_{k,2^k}$ and H_k are isomorphic for any $k \leq 3$. For all $k \geq 4$, we know that $W_{k,2^k}$ and H_k are not isomorphic (cf. [14] and Proposition 5). Hence, the embedding we propose here is optimal with respect to dilation.
Suppose that $k \geq 4$, and let us embed H_k into $W_{k,2^k}$. We suppose here that the vertices of H_k are the integers form 0 to $2^k - 1$. We consider the bijection f from $V(H_k)$ to $V(W_{k,2^k})$ defined as follows :

- $f(i) = (1, i)$ for all $i \in [0; 2^{k-1} - 1]$;
- $f(i) = (2, i - 2^{k-1})$ for all $i \in [2^{k-1}; 2^k - 1]$.

An example of such an embedding, with $k = 4$, is illustrated in Figure 9.

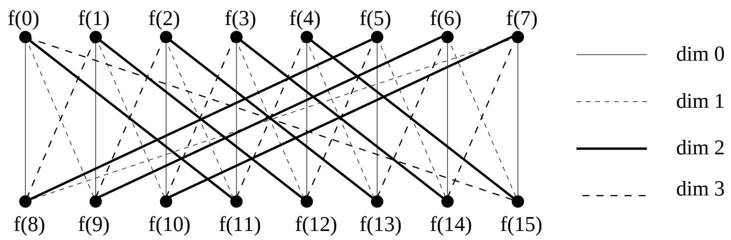

Fig. 9. Embedding of H_4 into $W_{4,16}$

Let us consider a vertex u in H_k. For any neighbour v of u, we compute the distance $d(f(u), f(v))$ in $W_{k,2^k}$. For this, we distinguish two cases :

- $0 \leq u \leq 2^{k-1} - 1$, i.e. the most weighted bit of $B(u)$ (where $B(u)$ is the binary representation of u) is equal to 0. In that case, $f(u) = (1, u)$. Then, if $v = 2^{k-1} + u$, we have $f(v) = (2, u)$, and the distance $d(f(u), f(v))$ in $W_{k,2^k}$ is equal to 1. If $v \neq 2^{k-1} + u$, this means that the most weighted bit of $B(v)$ is equal to 0, and there exists a $0 \leq l \leq k - 2$ such that $v = u \pm 2^l$. The distance from $f(u) = (1, u)$ to $f(v) = (1, u \pm 2^l)$ in $W_{k,2^k}$ is then equal

to 2 : indeed, $(1, u)$ is neighbour of $(2, u+2^l-1)$, which is itself neighbour of $(1, u+2^l)$. On the other hand, $(1, u)$ is neighbour of $(2, u-1)$ (in dimension $k-1$), which is itself neighbour of $(1, u-2^l)$.
- When $2^{k-1}-1 \leq u \leq 2^k$ (i.e., the most weighted bit of $B(u)$ is equal to 1), we proceed the same way. By similar arguments, we can show that the dilation is also equal to 2 in that case.

□

Proposition 14 *For all $k \geq 3$, it is possible to embed the Knödel graph $W_{k,2^k}$ into the recursive circulant graph $G(2^k, 4)$ with dilation 3.*

Proof : Suppose that $k \geq 3$ and let us embed $W_{k,2^k}$ into $G(2^k, 4)$. In $G(2^k, 4)$, the vertices are the integers from 0 to 2^k-1. Let us then consider the bijection f from $V(W_{k,2^k})$ to $V(G(2^k, 4))$, defined as follows :

- $f((1, i)) = 2i$ for all $i \in [0; 2^{k-1}-1]$;
- $f((2, i)) = 2i+1$ for all $i \in [0; 2^{k-1}-1]$.

An example of such an embedding, with $k = 4$, is given in Figure 10.

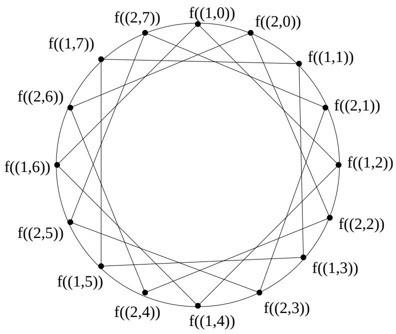

Fig. 10. Embedding of $W_{4,16}$ into $G(16, 4)$

Let us show that the dilation of such an embedding is equal to 3. For this, we show that for any vertex u of $W_{k,2^k}$ and for any neighbour v of u, we have $d(f(u), f(v)) \leq 3$ in $G(2^k, 4)$. Let us then distinguish two cases :

- $u = (1, i)$. In that case, $f(u) = 2i$ and the neighbours v of u in $W_{k,2^k}$ are of the form $(2, i+2^l-1 \bmod 2^{k-1})$ with $0 \leq l \leq k-1$. If $l = 0$, $v = (2, i)$ and $f(v) = 2i+1$ is neighbour of $f(u)$ in $G(2^k, 4)$. If $l \neq 0$, then $v = (2, i+2^l-1 \bmod 2^{k-1})$, and $f(v) = 2i+2^{l+1}-1 \bmod 2^k$. Depending on the parity of l, the distance $d(f(u), f(v))$ is either equal to 2 or 3. Indeed, $2i$ and $2i-1$ are neighbours. If $l+1 = 2q$ is even, then $2i-1$ and $2i-1+4^q$ are neighbours. Otherwise, that is if $l+1 = 2q+1$ is odd, we see that $f(v) = 2i-1+4^q+4^q$, which is at distance 3 from $f(u)$.
- $u = (2, i)$. In that case $f(u) = 2i+1$, and the neighbours v of u are of the form $(1, i-2^l+1 \bmod 2^{k-1})$ with $0 \leq l \leq k-1$. Using similar arguments as in the previous case, we can show here that the dilation is equal to 3.

□

Proposition 15 *For all $k \geq 3$, it is possible to embed the recursive circulant graph $G(2^k, 4)$ into $W_{k,2^k}$ with dilation 2.*

Proof : We recall that $W_{k,2^k}$ and $G(2^k, 4)$ are isomorphic for all $k \leq 2$. For all $k \geq 3$, we know that those two graphs are not isomorphic (cf. [14] and Proposition 5). Hence, the embedding proposed here is optimal with respect to dilation.
In $G(2^k, 4)$, the vertices are the integers from 0 to $2^k - 1$. We then consider the bijection f from $V(G(2^k, 4)$ to $V(W_{k,2^k})$, defined as follows :

- $f(2i) = (1, i)$ for all $i \in [0; 2^{k-1} - 1]$;
- $f(2i + 1) = (2, i)$ for all $i \in [0; 2^{k-1} - 1]$.

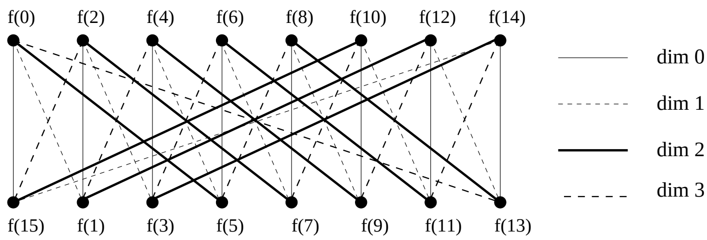

Fig. 11. Embedding of $G(16, 4)$ into $W_{4,16}$

Figure 11 illustrates such an embedding, in the case $k = 4$.
In order to show that the dilation of such an embedding is also equal to 2, let us show that for any vertex u of $G(2^k, 4)$ and for any neighbour v of u, we have $d(f(u), f(v)) \leq 2$ in $W_{k,2^k}$. For this, let us distinguish two cases :

- $u = 2m$ is even. In that case, $f(u) = (1, m)$ and the neighbours v of u in $G(2^k, 4)$ are of the form $u \pm 4^l \bmod 2^k$, with $0 \leq l \leq k - 1$. If $l = 0$, then $v = 2m \pm 1$. In both cases ($v = 2m+1$ and $v = 2m-1$), we see that $f(u)$ and $f(v)$ are neighbours in $W_{k,2^k}$. Now, if $l \neq 0$, $v = 2m \pm 4^l \bmod 2^k$ is even, and consequently $f(v) = (1, m \pm 2^{2l-1} \bmod 2^{k-1})$. In that case, we see that $f(v)$ is at distance 2 from $f(u)$ in $W_{k,2^k}$, because vertex $(2, m - 1)$ (resp. vertex $(2, m + 2^{2l-1} - 1 \bmod 2^{k-1})$ is neighbour of both vertices $f(u) = (1, m)$ and $(1, m - 2^{2l-1} \bmod 2^{k-1})$ (resp. and $(1, m + 2^{2l-1} \bmod 2^{k-1})$).
- If $u = 2m + 1$ is odd, we can show, by similar arguments, that the dilation is also equal to 2.

□

Park and Chwa [33] have also studied several embeddings between H_k, $G(2^k, 4)$ and $G(2^k, 2)$. The Propositions we gave here complete this study with embeddings among $W_{k,2^k}$, H_k, $G(2^k, 2)$ and $G(2^k, 4)$. Note that our main goal was to find embeddings with constant (and sometimes optimal) dilation, but that we have not considered the minimization of the congestion of these embeddings. Table 4 summarizes the results obtained here and in [33]. In the "Dilation" column, the asterisk ($*$) indicates that the result is optimal.

Embedding of	Into	Dilation	Reference
$G(2^k, 2)$	$G(2^k, 4)$	2^*	[33]
$G(2^k, 2)$	H_k	2^*	[33]
$G(2^k, 2)$	$W_{k,2^k}$	2^*	Prop. 11(b)
$G(2^k, 4)$	$G(2^k, 2)$	1^*	[33]
$G(2^k, 4)$	H_k	2^*	[33]
$G(2^k, 4)$	$W_{k,2^k}$	2^*	Prop. 15
H_k	$G(2^k, 2)$	1^*	[33]
H_k	$G(2^k, 4)$	2^*	[33]
H_k	$W_{k,2^k}$	2^*	Prop. 13
$W_{k,2^k}$	$G(2^k, 2)$	2	Prop. 11(a)
$W_{k,2^k}$	$G(2^k, 4)$	3	Prop. 14
$W_{k,2^k}$	H_k	4	Prop. 12

Table 4
Summary of the results : embeddings among $W(2^k, 2)$, H_k, $G(2^k, 4)$ and $G(2^k, 2)$

Properties	H_k	$G(2^k, 4)$	$W_{k,2^k}$
Number of vertices	2^k	2^k	2^k
Vertex-connectivity	k	k	$\in]\frac{2k}{3} ; k]$
Edge-connectivity	k	k	k
Degree	k	k	k
Diameter	k	$\lceil \frac{3k-1}{4} \rceil$	$\lceil \frac{k+2}{2} \rceil$
Vertex-transitivity	Yes	Yes	Yes
Edge-transitivity	Yes	No	No
Spanning subgraph Hamiltonian cycle	Yes	Yes	Yes
Spanning subgraph Binomial Tree	Yes	Yes	Yes

Table 5
Comparison between H_k, $G(2^k, 4)$ and $W_{k,2^k}$

6 Conclusion

In this paper, we have provided a survey of the known properties of Knödel graphs, mainly in terms of broadcasting and gossiping. To this survey, we have added some new results, that essentially deal with graph-theoretical properties of Knödel graphs. This study begins with the "general" Knödel graph $W_{\Delta,n}$, and then focuses more particularly on $W_{k,2^k}$.
Finally, we have provided some elements of comparison between the hypercube of dimension k, H_k, the recursive circulant graph $G(2^k, 4)$ and $W_{k,2^k}$. This comparison mainly includes embedding one architecture into another.

The study of the Knödel graphs, and more particularly of $W_{k,2^k}$, was motivated by the fact that these graphs had good properties concerning gossiping and broadcasting, as recalled in Section 3. Though it seems quite difficult to get general results concerning $W_{\Delta,n}$ for any Δ and n, the study of the Knödel graphs $W_{k,2^k}$ proves to be very instructive.
Thanks to this extended study of Knödel graphs, we hope to have given an overlook of many of the properties of this family which allows to understand better its structure and behaviour, in terms of communication as well as graph-theoretically speaking.

However, there remains many unanswered questions about this family. In particular, we have determined many properties in the case $W_{k,2^k}$, but several others remain undetermined in the general case. Among others, determining the diameter of the general Knödel graph $W_{\Delta,n}$ (for any even $n \geq 2$ and $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$) seems to be one of the most challenging problems.

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