

On Star Coloring of Graphs

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Abstract

In this paper, we deal with the notion of star coloring of graphs. A *star coloring* of an undirected graph G is a proper vertex coloring of G (i.e., no two neighbors are assigned the same color) such that any path of length 3 in G is not bicolored.

We give the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs and 2-dimensional grids. We also study and give bounds for the star chromatic number of other families of graphs, such as hypercubes, tori, d -dimensional grids, graphs with bounded treewidth and planar graphs.

Keywords : graphs, vertex coloring, proper coloring, star coloring, acyclic coloring, treewidth.

1 Introduction

All graphs considered are undirected. In the following definitions (and in the whole paper), the term *coloring* will be used to define *vertex coloring* of graphs. A **proper** coloring of a graph G is a labeling of the vertices of G such that no two neighbors in G are assigned the same label. Usually, the labeling (or coloring) of vertex x is denoted by $c(x)$. In the following, all the colorings that we will define and use are proper colorings.

Definition 1 (Star coloring) *A star coloring of a graph G is a proper coloring of G such that no path of length 3 in G is bicolored.*

We also introduce here the notion of *acyclic coloring*, that will be useful for our purpose.

Definition 2 (Acyclic coloring) *An acyclic coloring of a graph G is a proper coloring of G such that no cycle in G is bicolored.*

For any of the above colorings, we define by **chromatic number** of a graph G the minimum number of colors which are necessary to color G according to the definition of the given coloring. Depending on the coloring, the notation for the chromatic number of G differs ; it is denoted $\chi(G)$ for the *proper* coloring, $\chi_s(G)$ for the *star* coloring and $a(G)$ for the *acyclic* coloring.

By extension, the chromatic number of a family \mathcal{F} of graphs is the minimum number of colors that are necessary to color any member of \mathcal{F} according to the definition of the given coloring. Depending on the coloring, this will be denoted either by $\chi(\mathcal{F})$, $\chi_s(\mathcal{F})$ or $a(\mathcal{F})$.

The purpose of this paper is to determine and give properties on $\chi_s(\mathcal{F})$ for a large number of families of graphs. In Section 2, we motivate the problem and present general properties for

the star chromatic number of graphs. In the following sections (Section 3 to 6), we determine precisely $\chi_s(\mathcal{F})$ for trees, cycles, complete bipartite graphs and 2-dimensional grids and we give bounds on $\chi_s(\mathcal{F})$ for other families of graphs, such as hypercubes, graphs with bounded treewidth, outerplanar graphs, d -dimensional grids, 2-dimensional tori.

2 Generalities

We note that for any graph G , a star coloring of G is also an acyclic coloring of G : indeed, a cycle in G can be bicolored if and only if it is of even length, that is of length greater than or equal to 4. However, by definition of a star coloring, no path of length 3 in G can be bicolored. Thus the following observation.

Observation 1 *For any graph G , $a(G) \leq \chi_s(G)$.*

Actually we can remark that the star coloring is an acyclic coloring such that if we take two classes of colors then the induced subgraph is a bipartite graph composed only of stars. Star coloring was introduced in 1973 by Grünbaum [Grü73]. He linked star coloring to *acyclic coloring* by showing that any planar graph has an acyclic chromatic number less than or equal to 9, and by suggesting that this implies that any planar graph has a star chromatic number less than or equal to $9 \cdot 2^8 = 2304$.

However, this property can be generalized for any given graph G , as mentioned in [BKW99, BKRS00]. In [BKW99, BKRS00], the result was just stated, but no proof was given. We detail the proof here for completeness.

Theorem 1 (Relation acyclic/star coloring) *For any graph G , if the acyclic chromatic number of G satisfies $a(G) \leq k$, then the star chromatic number of G satisfies $\chi_s(G) \leq k \cdot 2^{k-1}$.*

In order to prove the theorem, we shall use the following proposition.

Proposition 1 *Let T be a tree and V_1 and V_2 be the bipartition of its set of vertices, then there exists a star coloring of T $c : V(T) \rightarrow \{0, 1, 2, 3\}$ such that if $v \in V_1$ then $c(v) \in \{0, 2\}$ and if $v \in V_2$ then $c(v) \in \{1, 3\}$.*

Proof : Let v be any vertex belonging to V_1 , we give the following coloring c of the vertices of T : $c(v) = 0$ and for $x \in V(T) \setminus \{v\}$, $c(x) = d(v, x) \bmod 4$. It is easy to see that this is a star coloring and that it has the required properties of the proposition. \square

Now we prove the theorem.

Proof : Let $G = (V, E)$ be a graph with $a(G) \leq k$, and let V_1, V_2, \dots, V_k be the color classes of an acyclic coloring of the vertices of G with k colors. The color classes form a partition of the set V and for any $i \neq j$ belonging to $\{1, \dots, k\}$, the subgraph induced by $V_i \cup V_j$, denoted by $G[V_i \cup V_j]$, is a forest. For any $G[V_i \cup V_j]$ ($i < j$), we denote by $c_{i,j}$ a star coloring of the vertices with 4 colors such that if $v \in V_i$ then $c_{i,j}(v) \in \{0, 2\}$, and if $v \in V_j$ then $c_{i,j}(v) \in \{1, 3\}$. We define the following coloring c of the vertices of G : let $u \in V$, then there is a unique $i \in \{1, \dots, k\}$ such that $v \in V_i$, let $c(u) = (c_{1,i}(u), \dots, c_{i-1,i}(u), *, c_{i,i+1}(u), \dots, c_{i,k}(u))$. We have to notice that the terms before the star belong to $\{0, 2\}$ and that the terms after the star belong to $\{1, 3\}$. By this definition we use at most $k2^{k-1}$ colors. It is clear that this is a proper coloring, indeed if $xy \in E$ then x and y belong to different sets V_i , and by definition $c(x) \neq c(y)$. Now we have to prove that there is no bicolored path of length 3. By contradiction let us assume that there is such a path P with $V(P) = \{u, v, w, t\}$ and $E(P) = \{uv, vw, wt\}$. By definition of the coloring the bicolored path is between V_i and V_j for some $i < j$. W.l.o.g., assume that $u, w \in V_i$ and $v, t \in V_j$; if $c(u) = c(w)$ and $c(v) = c(t)$, then we have $c_{i,j}(u) = c_{i,j}(w)$ and $c_{i,j}(v) = c_{i,j}(t)$, which is impossible because there is no bicolored path in $G[V_i \cup V_j]$ with the coloring $c_{i,j}$. This completes the proof. \square

In the following, let \mathcal{P} denote the family of planar graphs. We deduce the following easy result.

Theorem 2 $7 \leq \chi_s(\mathcal{P}) \leq 80$.

Proof : Grünbaum showed that any planar graph has an acyclic coloring using at most 9 colors, but conjectured that the exact answer was 5. Moreover, he gave an example of a planar graph G for which $a(G) = 5$. This conjecture was solved in 1979 by Borodin [Bor79]. This result, combined with Theorem 1 gives the upper bound for the theorem.

Moreover, there exists a planar graph G_2 for which any star coloring needs 7 colors. This graph is shown in Figure 1 (right). Thanks to the computer, we know that $\chi_s(G_2) = 7$. \square

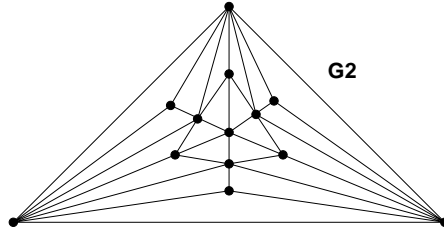


Figure 1: Planar graph : $\chi_s(G_2) = 7$

The girth g of a graph G is the length of its shortest cycle. In [BKW99], it is proved that if G is planar with girth $g \geq 5$ (resp. $g \geq 7$), then $a(G) \leq 4$ (resp. $a(G) \leq 3$). Together with Theorem 1, we deduce :

Corollary 1 *If G is a planar graph with girth $g \geq 5$, then $\chi_s(G) \leq 32$. If G is a planar graph with girth $g \geq 7$, then $\chi_s(G) \leq 12$.*

Several graphs are cartesian product of graphs (Grid, Hypercube, Tori), so it is interesting to have an upper bound for the star chromatic number of cartesian product of graphs. We recall that the cartesian product of two graphs $G = (V, E)$ and $G' = (V', E')$, denoted by $G \square G'$, is the graph such that the set of vertices is $V \times V'$ and two vertices (x, x') and (y, y') are linked if and only if $x = y$ and $x'y'$ is an edge of G' or $x' = y'$ and xy is an edge of G .

Theorem 3 *For any two graphs G and H , $\chi_s(G \square H) \leq \chi_s(G) \cdot \chi_s(H)$.*

Proof : Suppose that $\chi_s(G) = g$ and $\chi_s(H) = h$, and let c_G (resp. c_H) be a star coloring of G (resp. H) using g (resp. h) colors. In that case, we assign to any vertex (u, v) of $G \square H$ color $[c_g(u), c_h(v)]$. This coloring uses $g \cdot h$ colors, and this defines a star coloring. Indeed, suppose that there exists a path P of length 3 that is bicolored in $G \square H$, with $V(P) = \{x, y, z, t\}$ and $E(P) = \{xy, yz, zt\}$. Depending on the composition of the ordered pairs corresponding to the vertices of the path, we have 8 possible paths. We will only consider 4 of them, because by permuting the first and second component of each ordered pairs, we obtain the others. The 4 possible paths are :

- (1) $x = (u, v), y = (u, v_1), z = (u, v_2), t = (u, v_3)$
- (2) $x = (u, v), y = (u, v_1), z = (u, v_2), t = (u_4, v_2)$
- (3) $x = (u, v), y = (u, v_1), z = (u_2, v_1), t = (u_2, v_4)$
- (4) $x = (u, v), y = (u, v_1), z = (u_2, v_1), t = (u_5, v_1)$

Clearly, in the first case P cannot be bicolored, since the path v, v_1, v_2, v_3 is not bicolored in H . For the second case : y and t have different colors (v_1v_2 is an edge of H). For the third case : x and z have different colors (vv_1 is an edge of H). The same argument works for the last case. \square

Observation 2 For any graph G and for any $1 \leq \alpha \leq |V(G)|$, let G_1, \dots, G_p be the p connected components obtained by removing α vertices from G . In that case, $\chi_s(G) \leq \max_i \{\chi_s(G_i)\} + \alpha$.

Proof : Star color each G_i , and reconnect them by adding the α vertices previously deleted, using a new color for each of the α vertices. Any path of length 3 within a G_i will be star colored by construction, and if this path begins in G_i and ends in G_j with $i \neq j$, then it contains at least one of the α vertices, which has a unique color. Thus the path of length 3 cannot be bicolored, and we get a star coloring of G . \square

Remark 1 For any $\alpha \geq 1$, the above result is optimal for complete bipartite graphs $K_{n,m}$. W.l.o.g., suppose $n \leq m$ and let $\alpha = n$. Remove the $\alpha = n$ vertices of partition V_n . We then get m isolated vertices, which can be independently colored with a single color. Then, give a unique color to the $\alpha = n$ vertices. We then get a star coloring with $n + 1$ colors ; this coloring can be shown to be optimal by Theorem 6.

Observation 3 For any graph G that can be partitioned into p stables S_1, \dots, S_p , $\chi_s(G) \leq 1 + |V(G)| - \max_i \{|S_i|\}$.

Proof : Let $1 \leq j \leq p$ such that S_j is of maximum cardinality. Color each vertex of S_j with a single color c , and give new pairwise distinct different colors to all the others vertices. This coloring holds the desired number of colors. It is clearly a proper coloring, and it is also a star coloring, because there is only one color which is used at least twice. \square

Remark 2 The above result is optimal for complete p -partite graphs K_{s_1, s_2, \dots, s_p} .

3 Trees, Cycles, Complete Bipartite Graphs, Hypercubes

Theorem 4 Let \mathcal{F}_d be the family of forests such that d is the maximum depth over all the trees contained in \mathcal{F}_d . In that case, $\chi_s(\mathcal{F}_d) = \min\{3, d + 1\}$.

Proof : Let F be a forest contained in \mathcal{F}_d . When $d = 0$, the result is trivial (F holds no edge). When $d = 1$, we color each root of each tree contained in F with color 1, and all the remaining vertices by color 2. This is obviously a proper coloring, and since in that case there is no path of length 3, it is consequently a star coloring as well. Now we assume $d \geq 2$. We then color each vertex v , of depth d_v in F , as follows : $c(v) = d_v \bmod 3$. Clearly, this is a proper coloring of F and it is easy to see that it is a star coloring. \square

Theorem 5 Let C_n be the cycle with $n \geq 3$ vertices.

$$\chi_s(C_n) = \begin{cases} 4 & \text{when } n = 5 \\ 3 & \text{otherwise} \end{cases}$$

Proof : It can be easily checked that $\chi_s(C_5) = 4$. Now let us assume $n \neq 5$. Clearly, 3 colors at least are needed to star color C_n . We now distinguish 3 cases : first, if $n = 3k$, we color alternatively the vertices around the cycle by colors 1,2 and 3. Thus, for any vertex u , its two neighbors are assigned distinct colors, and consequently this is a valid star coloring. Hence $\chi_s(C_{3k}) \leq 3$. Suppose now $n = 3k + 1$. In that case, let us color $3k$ vertices of C_n consecutively, by repeating the sequence of colors 1,2 and 3. There remains 1 uncolored vertex, to which we assign color 2. One can check easily that this is also a valid star coloring, and thus $\chi_s(C_{3k+1}) \leq 3$. Finally, let $n = 3k + 2$. Since the case $n = 5$ is excluded here, we can assume $k \geq 2$. Thus $n = 3(k - 1) + 5$, with $k - 1 \geq 1$. In that case, let us color $3(k - 1)$ consecutive vertices along the

cycle, alternating colors 1,2 and 3. For the 5 remaining vertices, we give the following coloring : 2, 1, 2, 3, 2. It can be checked that this is a valid star coloring, and thus $\chi_s(C_{3k+2}) \leq 3$ for any $k \geq 2$. Globally, we have $\chi_s(C_n) = 3$ for any $n \neq 5$, and the result is proved. \square

Theorem 6 *Let $K_{n,m}$ be the complete bipartite graph with $n + m$ vertices. Then $\chi_s(K_{n,m}) = \min\{m, n\} + 1$.*

Proof : W.l.o.g., let $n \leq m$. The upper bound of $n + 1$ immediately derives from Observation 2 (cf. Remark 1 for a detailed proof).

Now let us prove that $\chi_s(K_{n,m}) \geq n + 1$. For this, let us show that any coloring with n colors will give us at least one bicolored cycle of length 4. Let S_n (resp. S_m) be the set of colors used to color the vertices of V_n (resp. V_m). Clearly, since all possible edges exist between vertices of V_n and vertices of V_m , we must have $S_n \cap S_m = \emptyset$ in order to achieve a proper coloring. Suppose then that we use $n - x$ colors for the vertices of V_n , and x colors for the vertices of V_m , with $1 \leq x \leq n - 1$. In that case, there exists at least 2 vertices a_n and b_n in V_n (resp. a_m and b_m in V_m) that are given the same color c_1 (resp. c_2). By definition of $K_{n,m}$, there exists a cycle of length 4 going through the vertices a_n, b_m, b_n, a_m , and this cycle is bicolored with colors c_1 and c_2 . Thus, no coloring that uses n colors can be a star coloring, and $\chi_s(K_{n,m}) \geq n + 1$. \square

For our purpose we introduce a new definition here : the k distance coloring of graphs.

Definition 3 (k distance coloring) *A k distance coloring of a graph G is a coloring of G such that any two vertices at distance at most k are assigned different colors.*

Remark 3 *For any $k \geq 1$, k distance coloring is a generalization of proper coloring, since the latter can be seen as a k distance coloring where $k = 1$.*

We define, as in Section 1, the k distance chromatic number of a graph G as being the minimum number of colors necessary to get a k distance coloring of G . It is noted $\chi_{\bar{k}}(G)$.

In this section, we will only focus on the case $k = 2$. Indeed, it is easy to see that for any graph G , a 2 distance coloring of G is also a star coloring of G . Hence the following observation.

Observation 4 *For any graph G , $\chi_s(G) \leq \chi_{\bar{2}}(G)$.*

2 distance coloring (and, more generally, k distance coloring) of hypercubes has already been studied [Wan97, KDP00]. The main result concerning 2 distance coloring is the following.

Theorem 7 [Wan97] *Let H_n denote the hypercube of dimension n . For any $n \geq 1$, $n + 1 \leq \chi_{\bar{2}}(H_n) \leq 2^{\lceil \log_2(n+1) \rceil}$.*

Wan conjectured that the above upper bound is tight. Note that Theorem 7 above implies that $\chi_{\bar{2}}(H_n)$ lies between $n + 1$ and $2n$, depending on the value of n ; this also shows that there is an infinite number of cases for which the result is optimal : indeed, when $n = 2^k - 1$ the lower and upper bounds coincide, and in that case $\chi_{\bar{2}}(H_{2^k-1}) = 2^k = n + 1$.

Thanks to this theorem, and thanks to Observation 4, we get the following property.

Property 1 *For any n , $\chi_s(H_n) \leq 2^{\lceil \log_2(n+1) \rceil}$.*

However, as can be seen in Table 1, this upper bound is not tight. All results given in bold characters in Table 1 are optimal ; the ones concerning star coloring are discussed below.

Property 2 $\chi_s(H_2) = 2$, $\chi_s(H_3) = 4$ and $\chi_s(H_4) = 5$.

Proof : The first result is trivial. It can also be easily seen that $\chi_s(H_3) \geq 4$. Since H_3 is a subgraph of H_4 , we deduce that $\chi_s(H_3) \geq 4$.

The upper bounds are given, in each case, by a star coloring with the appropriate number of colors. These are given in Figure 2. \square

n	$\chi_s(H_n)$	$\chi_{\bar{2}}(H_n)$	n	$\chi_s(H_n)$	$\chi_{\bar{2}}(H_n)$
2	3	4	5	6	8
3	4	4	6	6	8
4	5	8	7	$6 \leq \leq 8$	8

Table 1: Star and 2 distance coloring of H_n

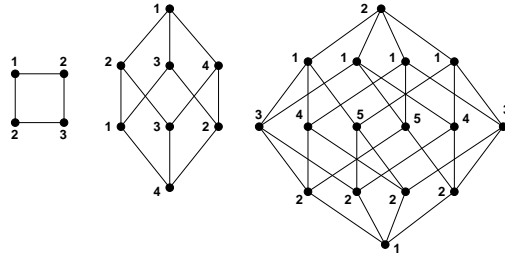


Figure 2: Star coloring of H_2 , H_3 and H_4

Property 3 $\chi_s(H_5) = 6$, $\chi_s(H_6) = 6$ and $\chi_s(H_7) \leq 8$.

Proof : The lower bound of 6 for $\chi_s(H_5)$ and $\chi_s(H_6)$ is given by computer : there is no possible star coloring of those two graphs with 5 colors.

The upper bounds are given in each of the 3 cases by an appropriate star coloring using the required number of colors. Those colorings are detailed in the appendix of [FRR01], where the result is given as a list of correspondences between each vertex and its color. \square

4 d -dimensional Grids

In this section, we study the star chromatic number of grids. More precisely, we give the star chromatic number of 2 dimensional grids, and we extend this result in order to get bounds on the star chromatic number of grids of dimension d .

We recall that the grid $G(n, m)$ is the cartesian product of two paths of length $n - 1$ and $m - 1$. Due to symmetries, we will always consider in the following that $m \geq n$. A summary of the results is given in Table 2 ; those results are detailed below.

	$m = 2$	$m = 3$	$m \geq 4$
$n = 2$	3	4	4
$n = 3$	xxx	4	4
$n \geq 4$	xxx	xxx	5

Table 2: Star coloring of 2-dimensional grids $G(n, m)$ ($n \leq m$)

Proposition 2 $\chi_s(G(2, 2)) = 3$, and for any $m \geq 4$, $\chi_s(G(2, m)) = \chi_s(G(3, m)) = 4$.

Proof : The first result is trivial.

It is easily checked that $\chi_s(G(2, m)) \geq 4$ for any $m \geq 3$, since star coloring of $G(2, 3)$ requires at least 4 colors, and since $G(2, 3)$ is subgraph of $G(2, m)$ for any $m \geq 3$. Moreover, as can be seen in Figure 3 (left), it is possible to find a 4 star coloring of $G(2, m)$. Thus $\chi_s(G(2, m)) \leq 4$, and altogether we have $\chi_s(G(2, m)) = 4$.

The fact that $\chi_s(G(3, m)) \geq 4$ for any $m \geq 3$ is trivial, since $G(2, 3)$ is a subgraph of $G(3, m)$. Moreover, Figure 3 (right) shows a 4 star coloring of $G(3, m)$ for any $m \geq 3$. \square

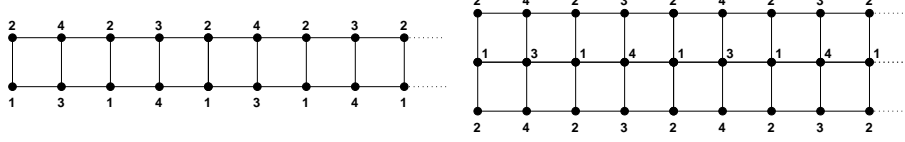


Figure 3: 4 star colorings of $G(2, m)$ (left) and $G(3, m)$ (right)

The main theorem of this section is the following.

Theorem 8 For any n and m such that $\min\{n, m\} \geq 4$, $\chi_s(G(n, m)) = 5$.

Proof : By a rather tedious case by case analysis (confirmed by the computer), it is possible to show that 5 colors at least are needed to color $G(4, 4)$. Hence, for any n and m such that $\min\{n, m\} \geq 4$, $\chi_s(G(n, m)) \geq 5$.

Now let us show that 5 colors are sufficient to get a star coloring of $G(n, m)$: for this, we will identify any vertex v_i of $G(n, m)$ by its coordinates (x_i, y_i) , $0 \leq x_i \leq n - 1$ and $0 \leq y_i \leq m - 1$. We now describe the coloring scheme :

- Any vertex $v = (x, y)$ with $x + y \equiv 0 \pmod 2$ is assigned color 1 ;
- Any vertex $v = (4p, 4q + 1)$ and $v = (4p + 2, 4q + 3)$ is assigned color 2 ;
- Any vertex $v = (4p, 4q + 3)$ and $v = (4p + 2, 4q + 1)$ is assigned color 3 ;
- Any vertex $v = (4p + 1, 4q)$ and $v = (4p + 3, 4q + 2)$ is assigned color 4 ;
- Any vertex $v = (4p + 1, 4q + 2)$ and $v = (4p + 3, 4q)$ is assigned color 5.

An example of this coloring is shown in Figure 4, with $n = 5$ and $m = 9$.

It can be easily checked that the above coloring assigns a color to each vertex, and that this coloring is proper ; moreover, this is a valid 5 star coloring of $G(n, m)$, since two vertices u and v at distance 2 are assigned the same color iff $c(u) = c(v) = 1$. In that case, there cannot exist a path of length 3 that is bicolored. Thus $\chi_s(G(n, m)) \leq 5$, and the result is proved. \square

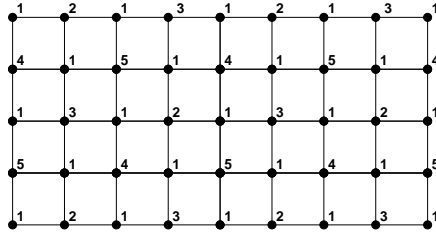


Figure 4: A 5 star coloring of $G(5, 9)$

We know that d -dimensional grids G_d are isomorphic to the cartesian product of d paths. Hence, by Theorems 3 and 4, we get an upper bound for $\chi_s(G_d)$: $\chi_s(G_d) \leq 3^d$ for any $d \geq 1$. Due to the result of Theorem 8 above, we can slightly improve this bound to $5 \cdot 3^{d-2}$, for any $d \geq 2$. However, it is even possible to do better : for this, we generalize the star coloring of paths and 2-dimensional grids. This is the purpose of the following theorem.

Theorem 9 Let G_d be any d -dimensional grid, $d \geq 1$. Then $\chi_s(G_d) \leq 2^d + 1$.

Proof : The idea here is to design a star coloring for G_d that generalizes the one given for $d = 2$ in Proof of Theorem 8. More precisely, we want to define a coloring of G_d such that, for any fixed $d - 2$ dimensions, the induced 2-dimensional grid has a coloring similar to the one of Figure 4. Let us now detail the coloring in itself : first, G_d being bipartite, we color every vertex of one of the two partitions of G_d by color 1. More formally, if every vertex $v \in G_d$ is defined by its d coordinates, that is $v = (x_1, x_2 \dots x_d)$, then $c(v) = 1 \forall x_1, x_2 \dots x_d$ s.t. $x_1 + x_2 + \dots + x_d \equiv 0 \pmod{2}$. Now, we need to assign the 2^d remaining colors to the remaining vertices. Again, we will do it depending on the coordinates $x_1, x_2 \dots x_d$ of the considered vertex v . For this, we will use a set S_d of 2^d vertices, chosen thanks to their coordinates, and for which every vertex of S_d will be assigned a unique color $2 \leq c \leq 2^d + 1$. Starting from the vertices of S_d we will then apply a rule to color the still uncolored vertices. First, let us detail the construction of S_d . Let V_0 be the set of vertices of G_d holding an even number (resp. odd number) of coordinates equal to 0 if d is odd (resp. if d is even). Let now $V_{0,1}$ be the subset of V_0 such that all the non-zero coordinates are all equal to 1. It is not difficult to see that $|V_{0,1}| = 2^{d-1}$.

Now let $V_{0,1,3}$ be the subset of V_0 such that all but one of the non-zero coordinates are equal to 1, the last one being equal to 3. Finally, let $V'_{0,1,3}$ be the maximum subset of $V_{0,1,3}$ such that any two vertices u and v in $V'_{0,1,3}$ have at least one zero-coordinate location which differs. In that case, it is clear that $|V'_{0,1,3}| = 2^{d-1}$. We now define S_d as follows : $S_d = V_{0,1} \cup V'_{0,1,3}$.

Example : in the case $d = 4$, $V_{0,1} = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 1)\}$, while $V'_{0,1,3} = \{(3, 0, 0, 0), (0, 3, 0, 0), (0, 0, 3, 0), (0, 0, 0, 3), (3, 1, 1, 0), (3, 1, 0, 1), (3, 0, 1, 1), (0, 3, 1, 1)\}$.

We assign to each vertex of S_d a unique color $2 \leq c \leq 2^d + 1$. Now, let us take a vertex $u \in S_d$ with color $c(u)$; we then assign $c(u)$ to any vertex u' obtained from u by any combination of the two following rules :

- **(Rule 1)** : add ± 4 to exactly one of the coordinates ;
- **(Rule 2)** : add ± 2 to exactly 2 of the coordinates.

In that case, it can be seen that all the vertices of the d -dimensional grid have been assigned a color : indeed, every vertex v such that the sum of its coordinates is even is assigned color 1 ; moreover, for every other vertex, S_d constitutes a basis of colored vertices, from which, thanks to Rules 1 and 2, every vertex in G_d is assigned a color.

This coloring is clearly a proper coloring of G_d (no two neighbors have been assigned the same color). Moreover, this is also a valid star coloring, since every vertex u s.t. $c(u) \neq 1$ is such that any vertex v with color $c(v) = c(u)$ is at least at distance 4 from u (by Rules 1 and 2). The above-mentioned coloring using $2^d + 1$ colors, we conclude that $\chi_s(G_d) \leq 2^d + 1$. \square

Remark 4 We note that for dimensions 1 and 2, the upper bound given by Theorem 9 for d -dimensional grids is tight (cf. Theorem 4 when $d = 1$ and Theorem 8 when $d = 2$).

5 2-dimensional Tori

In the following, for any $n, m \geq 3$, we denote the toroidal 2-dimensional grid by $TG(n, m)$. We recall that $TG(n, m)$ is the cartesian product of two cycles of length n and m . Our main result here is the following.

Theorem 10 For any $3 \leq n \leq m$, $5 \leq \chi_s(TG(n, m)) \leq 6$.

Proof : The proof is given in [FRR01]. More precisely, we show that there is an infinite number of cases of $TG(n, m)$, for which $\chi_s(TG(n, m)) = 5$. The fact that 5 colors are always needed, and that 6 colors always suffice for the star coloring of $TG(n, m)$, $n, m \geq 3$ derives from several propositions (see [FRR01]). \square

In some cases, we have been unable to determine precisely the number of colors necessary

to star color $TG(n, m)$, $n, m \geq 3$ (though we know it up to an additive constant of 1). However, we pose the following conjecture.

Conjecture 1 *For any $n \geq m \geq 3$*

$$\chi_s(TG(n, m)) = \begin{cases} 6 & \text{when } n = m = 3 \text{ or when } n = 3 \text{ and } m = 5 \\ 5 & \text{otherwise} \end{cases}$$

6 Graphs with Bounded Treewidth

The notion of treewidth was introduced by Robertson and Seymour [RS83]. A *tree decomposition* of a graph $G = (V, E)$ is a pair $(\{X_i | i \in I\}, T = (I, F))$ where $\{X_i | i \in I\}$ is a family of subsets of V , one for each node of T , and T a tree such that :

- (1) $\bigcup_{i \in I} X_i = V$
- (2) For all edges $vw \in E$, there exists an $i \in I$ with $v \in X_i$ and $w \in X_i$
- (3) For all $i, j, k \in I$: if j is on the path from i to k in T , then $X_i \cap X_k \subseteq X_j$

The *width* of a tree decomposition $(\{X_i | i \in I\}, T = (I, F))$ is $\max_{i \in I} |X_i| - 1$. The *treewidth* of a graph G is the minimum width over all possible tree decomposition of G .

We will prove the following theorem.

Theorem 11 *If a graph G has a treewidth at most k , then $\chi_s(G) \leq k(k+3)/2 + 1$*

Actually we will prove Theorem 11 for k -trees, because it is well known that the treewidth of a graph G is at most k ($k > 0$) if and only if G is a partial k -tree [Bod98].

We recall the definition of a k -tree [BLS99] :

- (1) a clique with k -vertices is a k -tree
- (2) If $T = (V, E)$ is a k -tree and C is a clique of T with k vertices and $x \notin V$, then $T' = (V \cup \{x\}, E \cup \{cx : c \in C\})$ is a k -tree.

If a k -tree holds exactly k vertices, then it is a clique by definition. If not, it contains at least a $k+1$ clique ; moreover, it is easy to see that the greedy coloring with $k+1$ colors of a k -tree gives an acyclic coloring. Hence we can deduce the following observation.

Observation 5 *For any $k \geq 1$ and any k -tree T_k :*

- $\chi_s(T_k) = k$ if $|V(T_k)| = k$;
- $k+1 \leq \chi_s(T_k) \leq k \cdot 2^{k-1}$ otherwise.

Theorem 11 is in fact a corollary of the following result, which gives a much tighter bound than the one of Observation 5.

Theorem 12 $k+1 \leq \chi_s(T_k) \leq k(k+3)/2 + 1$

Proof : Consider a k -tree G . We recall that a k -tree G is an intersection graph [MM99] and can be represented by a tree T and a subtree S_v for each v in G s.t. :

- (1) $uv \in E(G) \iff S_u \cap S_v \neq \emptyset$
- (2) for any $t \in T$, $|\{v : t \in S_v\}| = k+1$

We can see that by considering the tree decomposition of the k -tree. The tree T is the one of the tree decomposition and the subtree S_v for $v \in V(G)$ is exactly the subtree of T containing the nodes of T corresponding to the subsets of the tree decomposition containing v .

We root T at some node r and for each vertex v of G let $t(v)$ bet the first node of S_v obtained when traversing T in pre-order (i.e. $t(v)$ is the “highest” node of S_v). We choose some fixed preorder and order the nodes as v_1, v_2, \dots, v_n so that for $i < j$, $t(v_i)$ is considered in the preorder before $t(v_j)$. We $k(k+3)/2 + 1$ color the nodes in this order.

For each i we let :

$$X_{v_i} = \bigcup_{S_{v_j} \ni t(v_i)} \{v_l \neq v_i : S_{v_l} \ni t(v_j)\}$$

We show now that $|X_{v_i}| \leq k(k+3)/2$ and for any $v_j \in X_{v_i}$, $j < i$. Indeed let $A = \{a_1, a_2, \dots, a_k, v_i\}$ be the subset of vertices corresponding to $t(v_i)$. We assume that $t(a_i)$ is before $t(a_{i+1})$ ($i \in \{1, \dots, k-1\}$) in the preorder. We first give an upper bound for the number of subtrees S_{v_l} which contain $t(a_i)$ ($v_l \notin A$). The number of S_{v_l} ($v_l \notin A$) for $i \in \{1, \dots, k\}$ which contain $t(a_1)$ is at most k , because the corresponding subset can intersect A only in a_1 . The number of S_{v_l} ($v_l \notin A$) which contain $t(a_2)$ is at most $k-1$, a_1 is in the subset corresponding to $t(a_2)$ and we do not count S_{a_1} . It is easy to see that the number of S_{v_l} ($v_l \notin A$) which contain $t(a_i)$ is at most $k+1-i$, because the subset corresponding to $t(a_i)$ contains $\{a_1, a_2, \dots, a_i\}$. In total we have at most $\sum_{i=1}^k i = k(k+1)/2$ subtrees S_{v_l} with $v_l \notin A$ containing $t(a_i)$ $i \in \{1, \dots, k\}$. Now we have to add the number of S_{a_i} , this gives $|X_{v_i}| \leq k(k+3)/2$, without counting v_i . We color v_i with any color not yet used on X_{v_i} . This clearly yields a proper coloring, indeed if xy is an edge of G then $S_x \cap S_y \neq \emptyset$ and either $t(x) \in S_y$ or $t(y) \in S_x$, hence by construction x and y have different colors. We claim it also yields a coloring with no bichromatic P_4 (a path of length 3) : assume the contrary, and let $\{x, y, z, w\}$ be this P_4 labelled so that

- (1) $t(x)$ is the first of $t(x), t(y), t(z), t(w)$ considered in the preorder,
- (2) x and y have the same color,
- (3) xz and yz are in $E(G)$.

We have to notice that x, z, y are not in the same clique K_{k+1} of the graph G corresponding to a node of T , because by construction this would imply that the colors are different. Now $S_x \cap S_y = \emptyset$, $S_z \cap S_x \neq \emptyset$, $S_z \cap S_y \neq \emptyset$, so by (1) $t(y)$ is in S_z . Further since $zx \in E(G)$, by (1) we have $t(z)$ is in S_x . So $x \in X_y$, contradicting the fact that x and y get the same color. \square

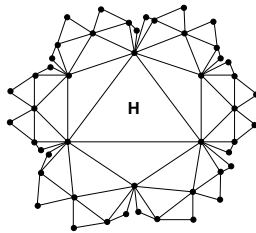


Figure 5: A 2-tree H that satisfies $\chi_s(H) = 6$

We can notice that for 1-trees (i.e. the usual trees), the upper bound we obtain matches the one given by Theorem 4. For 2-trees, the upper bound is optimal because of the graph in Figure 5, which has been shown by computer to have a star chromatic number equal to 6.

In the following, we will denote by \mathcal{O} the family of outerplanar graphs.

Corollary 2 $\chi_s(\mathcal{O}) = 6$.

Proof : It is well-known that any outerplanar graph is a partial 2-tree, then 6 colors suffice ; moreover the graph in Figure 5 which is also outerplanar, needs 6 colors. \square

7 Conclusion

In this paper, we have provided many new results concerning the star chromatic number of different families of graphs. In particular, we have provided exact results for paths and trees, cycles, complete bipartite graphs, 2-dimensional grids and outerplanar graphs. We have also determined bounds for the chromatic number in several other families of graphs, such as tori, d -dimensional grids, graphs with bounded treewidth, planar graphs and hypercubes. We have also determined several more general properties concerning the star chromatic number.

Using the techniques of [AMR90], we can show that the star chromatic number of a graph of maximum degree Δ is $O(\Delta^{\frac{3}{2}})$ and that for every Δ , there exists a graph of maximum degree Δ whose star chromatic number exceeds $\epsilon \cdot \frac{\Delta^{3/2}}{\log \Delta}$ for some positive absolute constant ϵ . Details will appear in the journal version of the paper.

A large number of problems remain open here, such as getting optimal results for other families of graphs, or refining our non optimal bounds ; getting one or several methods to provide good lower bounds for the star chromatic number is also another challenging problem.

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