Recognizing Recursive Circulant Graphs (Extended Abstract)

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Abstract

Recursive circulant graphs $G(N, d)$ have been introduced in 1994 by Park and Chwa [PC94] as a new topology for interconnection networks. Recursive circulant graphs $G(N, d)$ are circulant graphs with N nodes and with jumps of powers of d. A subfamily of recursive circulant graphs (more precisely, $G(2^k, 4)$) is of same order and degree than the hypercube of dimension k, with sometimes better parameters, such as diameter [PC94,GMR98]. Embeddings among recursive circulant graphs, hypercubes and Knödel graphs of order 2^k have also been studied in [PC,FR98b]. Here, following a question raised in [CFG99], we give, thanks to a sharp structural analysis of such graphs, an $O(cd^{m+2} \cdot (2m)^d)$ algorithm to determine if a given graph is a recursive circulant graph of the form $G(cd^m, d)$, for any $d \geq 3$, except in the case where c is even while d is odd. Notably, in the case where $d = O(1)$, this gives an $O(N \cdot (\log(N))^{O(1)})$ algorithm, with $N = cd^m$. Moreover, applying this algorithm to recursive circulant graphs $G(2^k, 4)$ gives us an $O(2^k \cdot k^4)$ recognition algorithm for such graphs.

Key words: Recursive circulant graphs, recognition, algorithm, complexity.

1 Introduction

Recursive circulant graphs $G(N, d)$ have been introduced in 1994 by Park and Chwa [PC94]. Since that time, recursive circulant graphs have been widely studied [Mic96,PC,GMR98,FR98a], and some more properties have been shown (Hamiltonian decomposition, edge-forwarding index, embeddings, etc.). More precisely, the subfamily of recursive circulant graphs of the form $G(2^k, 4)$ was presented by Park and Chwa as a new topology for multicomputer networks, because of its nice properties concerning their diameter, routing schemes, mean internode distance, etc. For instance, it has the same order and degree as the hypercube of dimension k, H_k, but has a smaller diameter : $\left\lceil \frac{3k-1}{4} \right\rceil$ compared to k for the hypercube.

Definition 1 [Recursive Circulant Graph $G(N, d)$ [PC94]]
The *recursive circulant graph* of order N is denoted $G(N, d)$. It has vertex set $V = \{0, 1, 2, \ldots, N-1\}$ and edge set $E = \{(v, w) \mid \exists\ i,\ 0 \leq i \leq \lceil \log_d N \rceil - 1,$ such that $v + d^i \equiv w \bmod N\}$.

In $G(N, d)$, any edge that connects a vertex v to a vertex $v \pm d^i \bmod N$, with $0 \leq i \leq \lceil \log_d N \rceil - 1$, is said to be *in dimension* i. Figure 1 shows two examples of recursive circulant graphs.

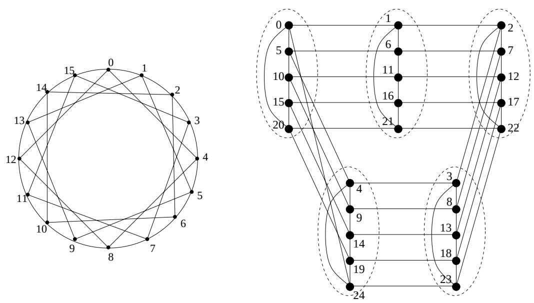

Fig. 1. $G(16, 4)$ (left) and $G(25, 5)$ (right)

Recursive circulant graphs $G(N, d)$ are Cayley graphs, and therefore are vertex-transitive [PC94]. They also are Hamiltonian decomposable [GMR98].
We will focus here on the case where $N = cd^m$, $1 \leq c < d$ is a multiple of a power of d. The recursive circulant graphs $G(cd^m, d)$ have been widely studied, notably in [Mic96,PC94]. For any $d \geq 3$, recursive circulant graphs $G(cd^m, d)$ are regular of degree $2m$ if $c = 1$, $2m + 1$ if $c = 2$ and $2m + 2$ otherwise. Moreover, $G(2^k, 4)$ have been shown to be Minimum Broadcast Graphs, Minimum Gossip Graphs and Minimum Linear Gossip Graphs [FR98a]. From this point of view, it is interesting to develop an algorithm which recognizes any recursive circulant graph of the form $G(2^k, 4)$. More generally, the question of the complexity of an algorithm which, for a given graph G, determines if G is isomorphic to the recursive circulant graph $G(N, d)$ is of interest. Here, we give an answer concerning this question for most of the recursive circulant graphs of the form $G(cd^m, d)$. More precisely, we answer the question for any $d \geq 3$, except in the case where c is even while d is odd.

2 Description of the Technique and Preliminary Results

Let us briefly explain the technique that we use : it consists in distinguishing in the given graph G, those edges that are possibly in dimension 0 in the recursive circulant graph $G(cd^m, d)$. Once this is done, it is easy to check if

G, of order cd^m, is isomorphic to $G(cd^m, d)$. Indeed, the edges in dimension 0 in $G(cd^m, d)$ form a Hamiltonian cycle by definition ; hence, once the edges in dimension 0 are distinguished, it suffices to relabel the vertices accordingly, from 0 to $N - 1 = cd^m - 1$. Once this is done, it suffices to check that any vertex $0 \leq v \leq N - 1$ is connected to every vertex of the form $v \pm d^i \bmod N$, for any $1 \leq i \leq \lceil \log_d N \rceil - 1$, as in the definition of a recursive circulant graph. Non surprisingly, the most difficult and time consuming part of the algorithm consists in isolating those edges in dimension 0. This is done thanks to a two-steps technique, which is as follows :

(1) **Step 1** : G being given, distinguish the edges that are possibly in the highest dimension in $G(cd^m, d)$ (dimension $m - 1$ when $c = 1$, dimension m otherwise). This will be done by different methods, depending on the parity of d and/or c.
(2) **Step 2** : we recursively distinguish the edges in dimension $m - i$ ($2 \leq i \leq m$ when $c = 1$, $1 \leq i \leq m$ otherwise) by calculating, in $G(cd^m, d)$, the number of pairwise distinct cycles of length $d + 1$ that go through an edge in dimension $m - i + 1$. This is done using the same routine m or $m - 1$ times (depending on the value of c), starting from the highest dimension (whose edges have been distinguished in **Step 1**), in order to reach dimension 0.

As mentioned above, most of the proofs here rely on the computation of the number of pairwise *distinct* cycles of fixed length that go through a particular edge in the graph, where we define two cycles to be *distinct* iff they differ on at least one edge. The main results are the following.

Property 1 [Step 1] For any $m \geq 1$, $d \geq 3$ and $1 \leq c < d$ (except in the case where c is even and d is odd), there exists an $O(cd^{m+2} \cdot (2m)^d)$ algorithm to realize **Step 1** of a recognition algorithm of $G(cd^m, d)$.

The technique we use for **Step 2** of the algorithm is the following : from any set $S_p \in E(G)$ of edges possibly in dimension p, to determine the set S_{p-1} of edges possibly in dimension $p - 1$. We start with $p = m - 1$ when $c = 1$, and $p = m$ when $c \geq 2$. For this, we take an edge e_p in S_p, search for cycles of length $d + 1$ that go through e_p and no other edge in S_q, $q \geq p$. We can prove that if such a cycle C exists, then all the edges distinct from e_p in C are in dimension $p - 1$, thus they belong to S_{p-1}. We repeat this till we reach dimension 0, from which we can construct the graph and check its isomorphism to $G(cd^m, d)$.

Property 2 [Step 2] For any $m \geq 1$, $d \geq 3$ and $1 \leq c < d$, there exists an $O(cd^{m+2} \cdot (2m)^d)$ algorithm realizing **Step 2** of a recognition algorithm of $G(cd^m, d)$.

Theorem 4 [Recognition of $G(cd^m, d)$**]** For any $d \geq 3$ and $m \geq 1$, there exists an $O(cd^{m+2} \cdot (2m)^d)$ algorithm to recognize any recursive circulant graph of the form $G(cd^m, d)$, except in the case when c is even while d is odd.

As a Corollary of Theorem 2, we give an $O(2^k \cdot k^4)$ algorithm to recognize recursive circulant graphs $G(2^k, 4)$. As indicated in the Introduction, it is interesting to get a recognition algorithm for those graphs, since $G(2^k, 4)$ competes well with the hypercube of dimension k and the Knödel graph of order 2^k, and since they have good broadcasting and gossiping properties.

Corollary 1 [Recognition of $G(2^k, 4)$] There exists an $O(2^k \cdot k^4)$ algorithm to recognize any recursive circulant graph of the form $G(2^k, 4)$.

We note that the above result is to be compared with the rather simple $O(2^k)$ algorithm to recognize the hypercube of dimension k, and the result of [CFG99], which gives an algorithm to recognize the Knödel graph $W_{k,2^k}$ in $O(2^k \cdot k^3)$.

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