# On the Oriented Chromatic Number of Grids

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**Abstract**

In this paper, we focus on the *oriented coloring* of graphs. Oriented coloring is a coloring of the vertices of an oriented graph $G$ without symmetric arcs such that (i) no two neighbors in $G$ are assigned the same color, and (ii) if two vertices $u$ and $v$ such that $(u,v) \in A(G)$ are assigned colors $c(u)$ and $c(v)$, then for any $(z,t) \in A(G)$, we cannot have simultaneously $c(z) = c(v)$ and $c(t) = c(u)$. The oriented chromatic number of an unoriented graph $G$ is the smallest number $k$ of colors for which *any* of the orientations of $G$ can be colored with $k$ colors.

The main results we obtain in this paper are bounds on the oriented chromatic number of particular families of planar graphs, namely 2-dimensional grids, fat trees and fat fat trees.

**Keywords:** graphs, oriented coloring, proper coloring.

## 1 Introduction

In this paper, we focus on a particular vertex coloring of graphs, called *oriented coloring*. It consists in coloring the vertices of an oriented graph $G$ without symmetric arcs given the two following rules :

- The coloring is *proper*, that is no two neighbors can be assigned the same color.
- For any $(u,v)$ and $(z,t) \in A(G)$, if $c(u) = c(t)$ then $c(v) \neq c(z)$.

The oriented chromatic number of an oriented graph $G$ (without opposite arcs) is the minimum number of colors needed to color $G$ according to the above rules, and is denoted $\vec{\chi}(G)$. The *oriented chromatic number* $\vec{\chi}(G)$ of an unoriented graph $G$ is then defined as the maximum oriented chromatic number over all the possible orientations of $G$.

The oriented chromatic number of a family $\mathcal{F}$ of graphs is defined to be the minimum number of colors that are necessary to color any member of $\mathcal{F}$ according to the above rules. This number is denoted by $\vec{\chi}(\mathcal{F})$.

In this paper, we focus on the oriented chromatic number of 2-dimensional grids, and some of its subgraphs. Concerning grids, surprisingly for such structured and "simple" graphs, exact results seem very difficult to extract. In this paper, we were only able to determine tight but non matching bounds on the oriented chromatic number of 2-dimensional grids. For some particular cases (2-dimensional grids where the number of rows is equal to 2 or 3), it is possible to get exact results. This also allows us to determine exact results for subgraphs of 2-dimensional grids, namely fat trees and fat fat trees.

The paper is organized as follows : in Section 2, we give bounds for the oriented chromatic number of the 2-dimensional grid, using homomorphism arguments ; in Section 3, we focus on some particular cases for 2-dimensional grids (number of rows being either 2 or 3) for which we give exact results, and we derive some other exact results for subgraphs of 2-dimensional grids, namely fat trees and fat fat trees.

## 2 2-Dimensional Grids

In this Section, we give bounds on the oriented chromatic number of 2-dimensional grids. In particular, the upper bound will be determined using homomorphism arguments. For this purpose, we first define formally the notion of homomorphism.

Given two oriented graphs $G = (V, A)$ and $G' = (V', A')$ a homomorphism from $G$ to $G'$ is any mapping $f : V \to V'$ satisfying

$$(x, y) \in A \implies (f(x), f(y)) \in A'.$$

The Oriented Coloring Problem can then be stated as follows : given an oriented graph $G = (V, A)$, find the smallest number of vertices of an oriented graph $G' = (V', A')$ for which $G \to G'$. We will then say that $G$ is colored by a homomorphism in $G'$.

It is easy to see that this smallest number is $\vec{\chi}(G)$ for any oriented graph $G$. The Oriented Coloring Problem has been extensively studied [RS94, NRS97, Sop97, BKN+98, BKN+99, NR99, BFK+01] these last years.

Using this notion, we get the following result.

**Theorem 1** $7 \leq \vec{\chi}(\mathcal{G}_2) \leq 11$.

**Proof** : The lower bound can be shown by a very tedious case by case analysis, which has also been confirmed by computer : the orientation $G^*$ of the grid $G(4, 5)$ that is given in Figure 1 is such that any oriented coloring of $G^*$ requires at least 7 colors (we note that the computer has given us many different orientations for which 7 colors are necessary). Hence, for any grid $G(m, n)$ such that $\min\{m, n\} \geq 4$ and $\max\{m, n\} \geq 5$, we have $\vec{\chi}(G(m, n)) \geq 7$.

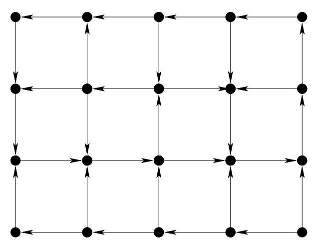

Figure 1: Any oriented coloring $G^*$ needs at least 7 colors

The upper bound comes from the fact that any orientation of $G(m, n)$ can be colored by a homomorphism in the graph $T(11; 1, 3, 4, 5, 9)$, or $T_{11}$, that is the oriented circulant graph with 11 vertices, and chords of jump 1,3,4,5 and 9. One of the properties, shown in [BKN+99], of $T_{11}$ that we are going to use here is the following :

**Property P1** : for any two distinct vertices of $T_{11}$, there exists at least two distinct paths of (unoriented) length 2 joining $u$ and $v$ for any given orientation of this path.

For any $m, n \geq 2$, we recall that we define any vertex $u$ by its coordinates $(i, j)$, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ ($i$ is the $x$-coordinate and $j$ the $y$-coordinate). Now let us color any given orientation of $G(m, n)$, by a homomorphism in $T_{11}$.

**Step 1 (First row)** : Clearly, it is always possible to color any oriented path, in such a way that homomorphism in $T_{11}$ is respected, since the indegree and outdegree of any vertex in $T_{11}$ is equal to 5 (actually, it is well known that 3 colors are enough to color any directed tree (and consequently any directed path)).

**Step 2 (Following rows)** : The method we are going to show for the second row also works for the following ones, and should then repeated for all the remaining ones.
First of all we color vertex $(1, 0)$ with a color different from the color of $(0, 1)$, which is always possible because the outdegree of any vertex of $T_{11}$ is 5. Then, using Property P1, it is always possible to color vertex $(1, 1)$ by a color different from the color of vertex $(0, 2)$, still by a homomorphism in $T_{11}$. Then we can continue this method to color the whole second row. Hence, it is always possible to color vertices of the $(r+1)$-th row, $r \geq 1$, once the vertices of the $r$-th row are colored, by a homomorphism in $T_{11}$.

Thus, there exists a way to color any orientation of $G(m, n)$ with 11 colors, and consequently $\vec{\chi}(G(m, n)) \leq 11$. □

Thanks to the computer, we have determined the oriented chromatic number of a large number of orientations of grids $G(m, n)$, for different values of $m$ and $n$, and we have been unable so far to find an orientation $G^*$ for which the number of colors necessary to color $G^*$ strictly exceeds 7. Moreover, for every orientation that has been fed to the computer (more than several tenth of millions of occurrences), we have shown that they all could be colored by a homomorphism in $T(7; 1, 2, 4)$ (something which we have not been able to prove theoretically so far). These results incline us to pose the following conjecture.

**Conjecture 1** *$\vec{\chi}(\mathcal{G}_2) = 7$, and any orientation of $\mathcal{G}_2$ can be colored by a homomorphism in $T(7; 1, 2, 4)$.*

## 3 "Small" 2-Dimensional Grids and some of its Subgraphs

Surprisingly, determining the exact oriented chromatic number in 2-dimensional grids seems to be a very difficult problem. This is why we now turn our attention to "smaller" cases (2-dimensional grids having either 2 or 3 rows), and to subgraphs of the 2-dimensional grids, such as fat trees and fat fat trees.

**Proposition 1**

- $\vec{\chi}(G(2, 2)) = 4$ ;
- $\vec{\chi}(G(2, 3)) = 5$ ;
- *For any* $n \geq 4$, $\vec{\chi}(G(2, n)) = 6$.

**Proof** : It is easy to find an orientation of $G(2, 2)$ for which any oriented coloring needs 4 colors. In the case $n = 3$, it can be easily seen that 5 colors are enough to color any orientation of $G(2, 3)$ : for this, consider the coloring shown in Figure 2 (left). In this coloring, vertices are assigned colors in such a way that an edge has its extremities colored by a unique pair of colors. Hence, for any orientation of $G(2, 3)$, this coloring will be an oriented coloring and thus $\vec{\chi}(G(2, 3)) \leq 5$.

Moreover, as shown in Figure 2 (right), there exists an orientation $G^*$ of $G(2,3)$ such that any oriented coloring of $G^*$ needs at least 5 colors. Thus $\vec{\chi}(G(2,3)) = 5$.

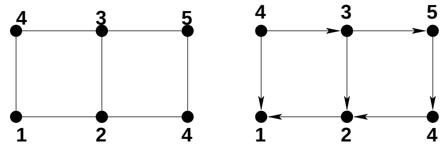


Figure 2: $\vec{\chi}(G(2,3)) \leq 5$ (left) $\vec{\chi}(G^*) \geq 5$ (right)

For the case $n \geq 4$, the lower bound derives from the study of $G(2,4)$, for which we can find an orientation $G^*$ such that any oriented coloring of $G^*$ requires at least 6 colors. Indeed, let $G^*$ be the oriented graph shown in Figure 3. It can be easily seen that any oriented coloring of $G^*$ requires at least 4 colors (for this, consider only the 4 leftmost vertices of the graph given in Figure 3). Now we will show that even 5 colors are not enough. Suppose we try to color $G^*$ with 5 colors : then $u_1$ can either be colored 1 or 5. If $c(u_1) = 1$, then $c(u_2) = 5$ and no color can be assigned to $u_3$. Thus, $c(u_1) = 5$. In that case, $c(u_2) = 2$, and consequently no color can be assigned to $u_4$. Hence, $G^*$ cannot be colored in 5 colors, and $\vec{\chi}(G(2,4)) \geq 6$.

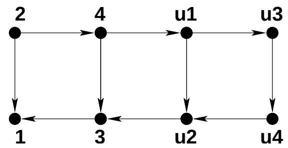


Figure 3: Any oriented coloring of $G^*$ needs at least 6 colors

Since $G(2,4)$ is subgraph of $G(2,n)$ for any $n \geq 4$, we get $\vec{\chi}(G(2,n)) \geq 6$ for any $n \geq 4$. The upper bound can be obtained by a homomorphism in the digraph $A_6$ given in Figure 4. This digraph has the property that for any two distinct vertices $u$ and $v$, there exists a path of (unoriented) length 3 that joins $u$ to $v$, whatever the orientations of the 3 arcs on the path from $u$ to $v$ (it is only a time consuming exercise to check all the possibilities for $i < j$ $(1 \leq i \leq 6$ and $1 \leq j \leq 6)$, and to see that the graph $A_6$ has the required property, which has also been confirmed by computer).

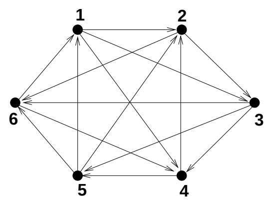


Figure 4: Digraph $A_6$

In that case, any grid $G(2,n)$, $n \geq 2$, can be colored by a homomorphism in $A_6$. Indeed, let $u_i = (0,i)$ and $v_i = (1,i)$, for any $0 \leq i \leq n-1$ in $G(2,n)$. $u_0$ and $v_0$ are connected by an arc. Whatever the orientation of this arc, we can color both $u_0$ and $v_0$ by a homomorphism in $A_6$. In that case $c(u_0) \neq c(v_0)$. $u_0$ and $v_0$ are also connected by a path of (unoriented) length 3, via $u_1$ and $v_1$. Thanks to the above property of $A_6$, it remains possible to color $u_1$ and $v_1$ by a homomorphism in $A_6$, such that $c(u_1) \neq c(v_1)$. Thus, we can repeat the previous step as many times as necessary, till we color $u_{n-1}$ and $v_{n-1}$. Hence, any orientation of $G(2,n)$ can be colored by a homomorphism in $A_6$. Since $|V(A_6)| = 6$, we get that $\vec{\chi}(G(2,n)) \leq 6$ for any $n \geq 2$, and altogether we have shown that $\vec{\chi}(G(2,n)) = 6$ for any $n \geq 4$. □

We note that the method described above can be extended from grids of the form $G(2,n)$ (or *ladders*) to *fat trees*. Fat trees are graphs generated by intersection of several horizontal ladders with several vertical ladders, in such a way that there do not exist two horizontal ladders $H_1$ and $H_2$ which both intersect two vertical ladders $V_1$ and $V_2$ (an example of fat tree is given in Figure 5).

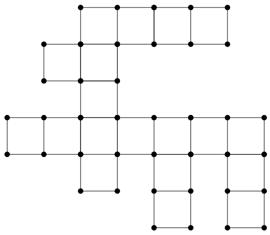

Figure 5: An example of fat tree

Hence, we get the following corollary.

**Corollary 1** *Let $\mathcal{F}_T$ be the family of fat trees. Then, $\vec{\chi}(\mathcal{F}_T) = 6$.*

Now, we turn to the case of grids of the form $G(3,n)$, $n \geq 3$. Let us denote by $\mathcal{C}_3$ the graph given in Figure 6.

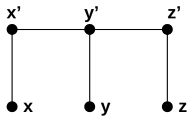


Figure 6: Graph $\mathcal{C}_3$

We then have the following lemma.

**Lemma 1** *For any orientation of $\mathcal{C}_3$, and any given coloring of vertices $x$, $y$ and $z$ among colors 0 to 6, it is possible to find a coloring of vertices $x'$,$y'$ and $z'$ that respects homomorphism into $T(7;1,2,4)$.*

**Proof** : We recall that there exists in $T(7;1,2,4)$ at least one path of length 2 between any two distinct vertices whatever the orientation of the arcs forming this path [BKN+99]. Since in $T(7;1,2,4)$, $d^+(x) = d^-(x) = 3$ for any vertex $x$, it is easy to see that for any two distinct vertices $u$ and $v$ of $T(7;1,2,4)$ there exits at least two paths $P_1 = (u, u_1, v_1, v)$ and $P_2 = (u, u_2, v_2, v)$ of (unoriented) length 3 for any orientation of the arcs forming this path, such that $v_1 \neq v_2$.

In that case, it is possible to find a path $P = (x, x', y', y)$ of length 3 joining $x$ to $y$, for any orientation of the arcs forming this path, by a homomorphism in $T(7;1,2,4)$, and such that $c(y') \neq c(z)$ in $\mathcal{C}_3$. Since there also exists in $T(7;1,2,4)$ at least one path of length 2 between any two distinct vertices, whatever the orientation of the arcs forming this path, it is possible to assign a color to $z'$ in $\mathcal{C}_3$ that respects homomorphism into $T(7;1,2,4)$. Thus $c(x')$, $c(y')$ and $c(z')$ are assigned, and homomorphism to $T(7;1,2,4)$ is respected. □

**Proposition 2**

- $\vec{\chi}(G(3,3)) = 6$ ;
- $\vec{\chi}(G(3,4)) = 6$ ;
- $\vec{\chi}(G(3,5)) = 6$ ;

- $6 \leq \vec{\chi}(G(3,n)) \leq 7$ *for any* $n \geq 6$.

**Proof** : For any $n \geq 4$, $G(2,4)$ is a subgraph of $G(3,n)$ ; since $\vec{\chi}(G(2,4)) = 6$, it follows that $\vec{\chi}(G(3,n)) \geq 6$ for any $n \geq 4$. In the case $n = 3$, it is possible to show that the digraph $G^*$ of Figure 7 cannot be colored with strictly less than 6 colors. Indeed, the 5 vertices $v_1$ to $v_5$ must be assigned pairwise different colors. Now suppose that we want to use only 5 colors ; then, $v_6$ can be assigned either color 4 or 5. If $c(v_6) = 4$, then $c(v_7) = 5$, $c(v_8) = 2$ and thus it is impossible to assign a color to $v_9$. Similarly, if $c(v_6) = 5$, then $c(v_7) = 1$, $c(v_8) = 1$ and thus it is impossible to assign a color to $v_9$. Since $G^*$ is a given orientation of $G(3,3)$, we conclude that $\vec{\chi}(G(3,3)) \geq 6$.

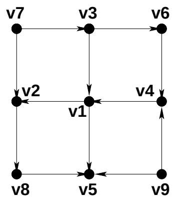


Figure 7: An orientation of $G(3,3)$ that needs 6 colors

Let us now show that 6 is an upper bound for $\vec{\chi}(G(3,3))$. First, if we do not have one of the two situations depicted in Figure 8 (left and middle), then 6 colors are enough (cf. Figure 8(right)).

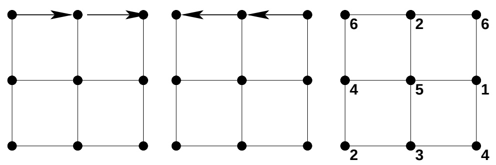


Figure 8: An oriented coloring (right) if the situations depicted (left and middle) do not appear

By symmetry (rotation of $G(3,3)$), we conclude that if at least one of the vertical or horizontal "borders" of $G(3,3)$ has not its two arcs oriented in the same direction, then 6 colors are always enough. Now, suppose two opposite borders are also in the same direction, as shown in Figure 9 (left) ; then it is still possible to find an oriented coloring with 6 colors (cf. Figure 9 (right)).

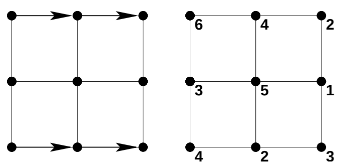


Figure 9: Oriented coloring (right) if the situation depicted (left) does not appear

Thanks to the symmetries, we then end up with only 2 remaining cases. They are depicted in Figure 10(left).

However, $G_1$ can be colored in 6 colors as shown in Figure 10(above, right) ; in $G_2$, we first assign the corners two different colors ($a$ and $b$) as shown in Figure 10(below, right). Moreover, whatever the orientations of the non already oriented arcs, we can show that there must exist 2 vertices $x$ and $y$ among the $v_i$s ($2 \leq i \leq 5$) such that :

- either arcs $(x, v_1)$ and $(y, v_1)$ are both ingoing arcs for $v_1$ ;
- or arcs $(v_1, x)$ and $(v_1, y)$ are both outgoing arcs for $v_1$.

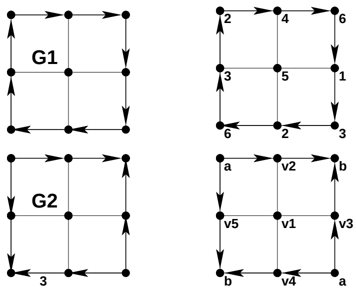


Figure 10: Last two possibilities

In both cases, $x$ and $y$ can be assigned the same color, and thus there remains 3 colors to assign to three vertices. In this situation, we conclude that 6 colors are enough to obtain an oriented coloring.

Altogether, we have shown that for any orientation of $G^*$ of $G(3,3)$, 6 colors suffice to color $G^*$, and thus $\vec{\chi}(G(3,3)) \leq 6$ ; hence, we finally have $\vec{\chi}(G(3,3)) = 6$.

In the cases $n = 4$ and $n = 5$, we have used the computer to show that for any orientation $G^*$ of $G(3,n)$, it is always possible to produce an oriented coloring of $G^*$ that uses 6 colors. Thus, $\vec{\chi}(G(3,4)) = 6$ and $\vec{\chi}(G(3,5)) = 6$.

We finish the proof with the case $n \geq 6$. Here, we proceed by a homomorphism in $T(7;1,2,4)$. Indeed, let $u_j = (0,j)$, $v_j = (1,j)$ and $w_j = (2,j)$ for any $0 \leq j \leq n-1$. First, we color $u_0$, $v_0$ and $w_0$, according to $T(7;1,2,4)$ (this is always possible). Now we use the result of Lemma 1 to color $u_1$, $v_1$ and $w_1$. If we repeat the same process $n-1$ times, we finally color every vertex of $G(3,n)$ in 7 colors, by a homomorphism in $T(7;1,2,4)$. This is feasible for any orientation of $G(3,n)$, thus $\vec{\chi}(G(3,n)) \leq 7$ for any $n$. □

As for Proposition 1, we note that the latter method can be extended from grids of the form $G(3,n)$ (or *fat ladders*) to *fat fat trees*, where fat fat trees are graphs generated from fat ladders, the same way as fat trees are generated from ladders (an example of a fat fat tree is given in Figure 11).

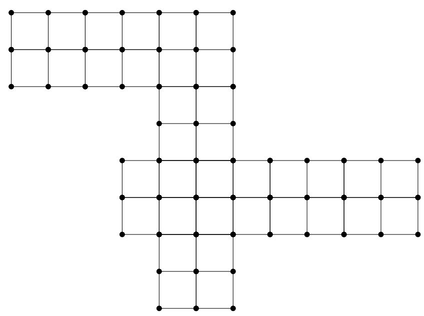

Figure 11: An example of fat fat tree

Hence, we get the following corollary.

**Corollary 2** *Let $\mathcal{FF}_T$ be the family of fat trees. Then,* $6 \leq \vec{\chi}(\mathcal{FF}_T) \leq 7$.

**Remark 1** *We note for completeness that we were able to prove, thanks to the computer, that $\vec{\chi}(G(4,4)) \leq 6$. Since $G(3,3)$ is a subgraph of $\vec{\chi}(G(4,n))$ for any $n \geq 3$, and since $\vec{\chi}(G(3,3)) = 6$ (cf. Proposition 2), we conclude that $\vec{\chi}(G(4,4)) = 6$.*

## 4 Conclusion

In this paper, we have studied the oriented coloring of 2-dimensional grids, as well as some subgraphs of 2-dimensional grids (namely, fat trees and fat fat trees). It may be surprising that the result is far from trivial : the oriented chromatic lies between 7 and 11, and tightening this bound seems to be a challenging problem. However, as mentioned in this paper, we conjecture that the actual answer is 7.

## 5 Acknowledgements

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