

A Study of Minimum Gossip Graphs

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Abstract

Gossiping and broadcasting are two problems of information dissemination described in a group of individuals connected by a communication network. In broadcasting, one individual has an item of information and needs to communicate it to everyone else. In gossiping, every person in the network knows a unique item of information and needs to communicate it to everyone else. Those two communication patterns find their main applications in the field of interconnection networks for parallel and distributed architecture.

After reviewing some of the main results that have been obtained on these problems, we give new properties concerning gossiping, mainly about the structure of the networks gossiping in minimum time and with a minimum number of communication links $G(n)$. These properties lead to new bounds for $G(n)$ in the general case, and in particular when $1 \leq n \leq 32$. Moreover, for some values of n (namely $n = 15$, $n = 24$ and $n = 28$), we show the exact value of $G(n)$.

1 Introduction

This paper presents new results on graphs performing gossiping in minimum time and with a minimum number of edges. Those new results mainly concern the structure of such graphs, and consequently give new bounds for their number of edges.

In the next section, we give some definitions introducing broadcasting and gossiping. In the third section, we give some other definitions and notations necessary to tackle the question. Section 4 provides a quick survey of broadcasting and gossiping. Section 5 is devoted to the structure of the concerned graphs, and the two following sections will provide respectively a series of upper and lower bounds concerning the number of edges of such graphs. Finally, section 8 presents new results and/or bounds for graphs having between 1 and 32 vertices, as well as a quick sum-up of the properties of minimum gossip graphs.

2 Definitions

We will use, as far as possible in the following, the same definitions and notations than those given in [dR94, HHL88, Lei92]. A parallel - or distributed - architecture will be modelled by a graph, where vertices will represent processors - or computers -, and edges will represent communication links. We shall consider a *constant time*, *1-port* and *full-duplex* model, e.g. :

- each message sent from one node to its neighbour takes one time unit ;
- each vertex can only communicate with one of its neighbours at the same time ;
- if an edge joins a vertex u to a vertex v , then the communication takes place both from u to v and from v to u .

Such a model is represented by undirected and connected graphs without loops or multiple edges.

2.1 Broadcasting

Broadcasting, also called *one-to-all*, refers to sending an item of information from one particular node to every other in the graph. The term $b(G)$ will denote the minimum amount of time necessary to broadcast in graph G from any vertex v , or the *broadcast time* of G . If one considers the complete graph K_n , it is quite obvious that $b(K_n) = \lceil \log_2 n \rceil$, since the number of informed vertices can at most double every time unit. Let b_n be this value of $b(K_n)$. A *broadcast graph* will denote any graph able to broadcast in b_n time units. It is not necessary though to consider K_n to get a broadcast graph. We then call a *Minimum Broadcast Graph* - or *MBG* - any broadcast graph with a minimum number of edges, which will be denoted by $B(n)$.

2.2 Gossiping

Gossiping, also called *all-to-all*, refers to the information dissemination problem that exists when each member of a set of n individuals knows a unique piece of information and must transmit it to every other person.

The term $g(G)$ will denote the minimum amount of time necessary to gossip in graph G , or the *gossip time* of G . First of all, it is obvious that $b(G) \leq g(G) \leq 2 \times b(G)$, since gossiping requires at least broadcasting, and since gossiping can be seen as broadcasting followed by a gathering of the information of each vertex. Moreover, Knödel [Kno75] has proved that the gossip time in the complete graph K_n is :

- $\lceil \log_2 n \rceil$ for n even ;
- $\lceil \log_2 n \rceil + 1$ for n odd.

Such a graph, able to gossip in $\lceil \log_2 n \rceil$ for n even (resp. in $\lceil \log_2 n \rceil + 1$ for n odd) is called a *gossip graph*, and g_n will denote its gossip time. As for broadcasting, it is not necessary to consider K_n to get a gossip graph. A *Minimum Gossip Graph* - or *MGG* - will then denote any gossip graph with a minimum number of edges. This number of edges is noted $G(n)$.

Remark : we will always use from now on the abbreviations “*MBG*” to refer to a minimum broadcast graph and “*MGG*” to refer to a minimum gossip graph.

3 Notations

3.1 General Notations

We call MBG_n (resp. MGG_n) a minimum broadcast graph (resp. minimum gossip graph) of order n .

Let A and B be two adjacent vertices of a graph G . AB will denote the edge joining A to B as well as the set of time units when this edge is used. We say that this set of time units *valuates* AB . $|AB|$ will denote the cardinality of the set AB , e.g. the number of communication steps between A and B during gossiping.

Similarly, A will denote the vertex A as well as the piece of information it contains before gossiping takes place.

We call *pendent edge* an edge AB joining a vertex A of degree 1 to the rest of the graph. Let B be of degree q . The pendent edge AB will be called a *(1,q)-type edge*.

We call *cut-edge* an edge AB such that its deletion in the graph disconnects it.

3.2 Notions of *MBT* and *RBT*

In this section, let us consider a graph G with n vertices. What we want to know is whether this graph allows gossiping in t time units or not. Most of the time, we know different characteristics of G , such as the degree(s) of one or several vertex (vertices) of G , or the valuation of some of its edges, or even both.

3.2.1 t -Maximum Broadcast Trees (*MBT*)

In order for G to gossip in t time units, it is necessary that each vertex of G is able to broadcast in t time units, especially vertices for which hypothesis are made.

We then call t -Maximum Broadcast Tree, or *MBT*, rooted in A , a tree rooted in A which allows to broadcast its information to a maximum number of vertices in t time units.

Example : Let's suppose that in a graph G of order 15, that we'd like to be MGG_{15} , we have a vertex A of degree 1 such that the edge AB joins A to the rest of the graph. Let's suppose that this edge is such that $AB = \{1, 5\}$, and that B is of degree 3. If we build the *MBT* rooted in A , we get the tree shown in Figure 1, which has 14 vertices. Consequently, under those hypothesis, A can't broadcast in 5 time units to the rest of the graph. Which means that at least one of the above hypothesis is wrong.

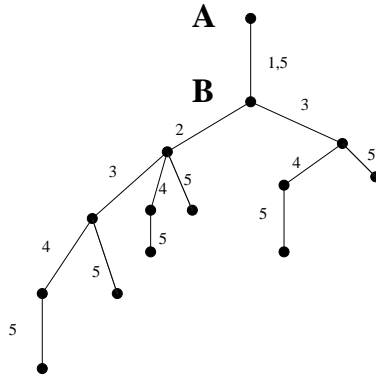


Figure 1: An example of *MBT*

3.2.2 Reverse Broadcast Trees (*RBT*)

In order for G to gossip within t time units, it is necessary that all the vertices of G can be informed of the content of each other vertex in t time units. This is also true for the vertices about which some hypothesis are made.

We then call *Reverse Broadcast Tree*, or *RBT* rooted in A , a tree rooted in A , which, following the hypothesis made about the vertices and edges of G , gathers in A the information of a maximum number of vertices in t time units. Moreover, if A_1A_2 is valued by $m_{A_1A_2}$ in the *MBT*, and by $r_{A_1A_2}$ in the corresponding *RBT*, we have the following equality : $r_{A_1A_2} + m_{A_1A_2} = t + 1$.

Remark : a requisite condition for a graph to be a MGG_n is that its *MBT* and *RBT* rooted in each of its vertices all have n or more vertices. However, this is obviously not a sufficient condition, because it doesn't take in account neither the number of edges of the graph nor the possible incompatibilities between the valuations in each *MBT* and *RBT*.

4 Known results

4.1 Immediate result

Property 1 For all n even, $G(n) \geq B(n)$.

Proof : Whatever the order of the graph is, gossiping requires broadcasting. Therefore, as gossip and broadcast times are the same for all n even, the number of edges in a *MGG* must be superior to the number of edges in a *MBG*.

4.2 Broadcasting

Property 2 [FHMP79] For all $n = 2^p$, $B(n) = p \times 2^{p-1}$.

Property 3 [DFE91] For all $n = 2^p - 2$ ($p \geq 3$), $B(n) = (p - 1) \times (2^{p-1} - 1)$.

4.3 Gossiping

Property 4 [Lab86] In a n -vertices tree T_n , the gossip time $g(T_n)$ is such that

$$g(T_n) \geq 2 \times \lceil \log_2 n \rceil - 1.$$

Property 5 [Lab93]

For all $n = 2^p$, $G(n) = p \times 2^{p-1}$.

For all $n = 2^p - 2$ ($p \geq 4$), $G(n) = (p - 1) \times (2^{p-1} - 1)$.

For all $n = 2^p - 4$ ($p \geq 6$), $G(n) = (p - 1) \times (2^{p-1} - 2)$.

5 Structure of a MGG_n

5.1 Cut-edges in a MGG_n

Theorem 1 In a MGG_n :

1. For all n even ($n \geq 4$), there is no cut-edge.
2. For all n odd ($n \geq 5$), the only cut-edges that can exist are pendent edges AB . Moreover, $AB = \{1, g_n\}$.

Proof : Let AB be a cut-edge dividing the graph G in G_1 and G_2 . Suppose that $|AB| = \{t_1, \dots, t_p\}$ where $1 \leq t_1 < t_2 \dots < t_p \leq g_n$. Let t_i (resp. t_j) be the smallest valuation on AB such that A (resp. B) knows all the information of G_1 (resp. G_2) within $t_i - 1$ (resp. $t_j - 1$) time units at most. Let $a = |V(G_1)|$ and $b = |V(G_2)|$, where $a + b = n$, and let us suppose, w.l.o.g., that $a \geq \lceil \frac{n}{2} \rceil$. Thanks to the broadcast times of each G_i , we have the four following inequalities :

- $t_i - 1 \geq \lceil \log_2 a \rceil$ and $g_n - t_i \geq \lceil \log_2 b \rceil$;
- $t_j - 1 \geq \lceil \log_2 b \rceil$ and $g_n - t_j \geq \lceil \log_2 a \rceil$.

Those inequalities lead to the following one : $g_n \geq \lceil \log_2 a \rceil + \lceil \log_2 b \rceil + 1$. **(I1)**

Now let n be even, and let us consider two cases :

- $b = 1$. In that case, $a = n - 1$ and **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n - 1) \rceil + 1$, which is false for any even $n \geq 4$.
- $2 \leq b \leq \frac{n}{2}$. In that case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 n \rceil + 1$, which is false for any n even.

Hence, if n is even ($n \geq 4$), there is no cut-edge in a *MGG*.

Let n be odd, and a and b as above. Let us distinguish three cases ;

- $b = 1$, that is BA is a pendent edge. Then **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2(n-1) \rceil$, which is true for any n . In that case, the four inequalities above lead to two cases :
 1. If $n \neq 2^p + 1$, we get $t_j \leq 1$ and $t_i \geq g_n$;
 2. If $n = 2^p + 1$, we get $t_j \leq 2$ and $t_i \geq g_n - 1$, which leads to four possible valuation schemes for the pendent edge BA . However, the *MBT*s and *RBT*s rooted in B hold enough vertices only in the case $BA = \{1, g_n\}$.
- $b = 2$. In that case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2(n-2) \rceil + 1$, which is true only for $n = 2^p + 1$. Let $V(G_1) = \{B, B_1\}$. Then BB_1 is a pendent edge and, as seen above, we have $BB_1 = \{1, g_n\}$. Moreover, the first four inequalities yield $t_j = 2$ and $t_i = g_n - 1$. However, the *MBT* rooted in B_1 in that case holds at most 2^p vertices for any $n \geq 5$. Hence, that case is impossible.
- $3 \leq b \leq \frac{n-1}{2}$. In that case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2(n+1) \rceil + 1$, which is false for any n .

Finally, if there is a cut-edge AB in a MGG_n where n is odd ($n \geq 5$), then it is a pendent edge ; moreover, $AB = \{1, g_n\}$.

Corollary 1 *In a MGG_n ($n \geq 4$) :*

- *there is no (1,2)-type edge ;*
- *two vertices of degree 1 cannot be adjacent to the same vertex.*

5.2 Vertices of degree 2 in a MGG_n

Theorem 2 *In a MGG_n :*

1. *For all n even ($n \geq 6$), if there is a vertex A of degree 2 with incident edges AB and AC , then 1 and g_n valuate AB or AC , e.g. $\{1, g_n\} \subseteq AB \cup AC$.*
2. *For all n odd ($n \geq 9$), if $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ and if there is a vertex A of degree 2 with incident edges AB and AC , then $\{1, g_n\} \subseteq AB \cup AC$.*

Proof : Let's suppose first that 1 doesn't valuate AB nor AC . Then the *MBT* rooted in A holds at most $3 \times 2^{g_n-3} + 1$ vertices. If n is even, we know that $n \geq 2^{g_n-1} + 2$. Subsequently, the *MBT* rooted in A doesn't hold enough vertices. If n is odd, we reach the same conclusion for all $n \geq 3 \times 2^{g_n-3} + 3$ (we recall that $g_n = p + 1$ when n is odd). In case $n = 3 \times 2^{g_n-3} + 1$, the *MBT* rooted in A holds exactly the necessary number of vertices, but the *RBT* doesn't for any $n \geq 9$.

Analogously, we prove that g_n valuates either AB or AC thanks to the *RBT* rooted in A .

5.3 Vertices of degree d in a MGG_n

Theorem 3 *For all n even such that $(2^d - 1) \times 2^{p-d} \leq n \leq 2^p$ with $d \leq p - 2$, there is no vertex of degree less or equal to d in a MGG_n .*

Proof : First, we recall that $g_n = p$ for such values of n . Suppose we have a vertex A of degree $d_1 \leq d$. Then the *MBT* rooted in A holds $(2^{d_1} - 1) \times 2^{g_n-d_1} + 1$ vertices, which is only possible for $d_1 = d$ and $n = (2^d - 1) \times 2^{g_n-d}$. In that case, though, we show that necessarily time units $1, 2 \dots d$ valuate the d edges incident to A . Analogously for the *RBT* rooted in A , time units $g_n, g_n - 1 \dots g_n - d + 1$ need to valuate those edges too. If we combine those two constraints, and as we supposed $g_n \geq d + 2$, we get a *MBT* rooted in A with two less vertices than needed. Therefore, there cannot be any vertex of degree less or equal to d in a MGG_n .

5.4 (1,3)-type edges in a MGG_n

Theorem 4 For all n odd ($n \geq 13$), if $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$, there is no (1,3)-type edge in a MGG_n .

Proof : Let us suppose we have a (1,3)-type edge AB , where A is the vertex of degree 1 and B_1 and B_2 the two other vertices adjacent to B . Theorem 1 yields that 1 and g_n valuate AB . Consequently, the MBT rooted in A holds $3 \times 2^{g_n-3} + 2$ vertices. In the case $n = 3 \times 2^{p-2} + 1$, however, some constraints on the valuations are given thanks to the MBT and RBT rooted in A (e.g. $2 \in BB_i$, $3 \in BB_j$ ($i \neq j$) and $(g_n - 1) \in BB_k$, $(g_n - 2) \in BB_l$ ($k \neq l$)). However, the MBT rooted in A according to these constraints holds at most $3 \times 2^{p-2} - 4$ vertices.

5.5 Adjacency of vertices of degree 2 in a MGG

Theorem 5 For all n even ($n \geq 8$ and $n \neq 2^{g_n-1} + 2$), two vertices of degree 2 can't be adjacent in a MGG_n .

Proof : Let's suppose we have two vertices of degree 2 adjacent, A and B . Theorem 2 yields that 1 and g_n valuate the edges adjacent to A and B , which leads to four possible valuation schemes. As previously, we determine some more constraints for each of those schemes such that the MBT rooted in A doesn't hold enough vertices.

Theorem 6 For all n odd, if $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, a vertex of degree 2 is adjacent to at least one vertex of degree 3 or more in a MGG_n .

Proof : Let's suppose we have three vertices of degree 2, A , B and C , such that A has B and C as neighbours. Theorem 2 yields that 1 and g_n valuate the edges adjacent to each of those vertices. Introducing first the valuation 1, we get a MBT rooted in A with $3 \times 2^{g_n-3} + 2$ vertices. Moreover, in the case $n = 3 \times 2^{g_n-3} + 1$, we introduce the valuation g_n and get to a MBT rooted in A with $3 \times 2^{g_n-3} - 4$ vertices.

6 Upper bounds for G_n

6.1 Immediate property

Theorem 7 For all n even, $G(n) \leq \frac{n \times g_n}{2}$; for all n odd, $G(n) \leq \frac{(n-1) \times g_n}{2}$.

Proof : Suppose first n is even. In order for an edge to be valuated by at least one time unit, each vertex is at worst of degree g_n . In that case, the corresponding MGG_n is g_n -regular and has $\frac{n \times g_n}{2}$ edges.

If n is odd, the proof is similar except that we know that, at each time unit, at least one vertex doesn't communicate with any other. So that $2 \times G(n) \leq n \times g_n - g_n$, hence the result.

6.2 Gossip graphs with $2n$ vertices

6.2.1 Knödel graphs

Definition 1 The Knödel graph [FP94] on $n \geq 2$ vertices (n even) and of maximum degree $\Delta \geq 1$ is denoted $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the couples (i, j) with $i=1,2$ and $0 \leq j \leq \frac{n}{2} - 1$. For every j , $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, j + 2^k - 1 \bmod \frac{n}{2})$, for $k = 0, \dots, \Delta - 1$.

For $0 \leq k \leq \Delta - 1$, an edge of $W_{\Delta,n}$ which connects a vertex $(1, j)$ to the vertex $(2, j + 2^k - 1 \bmod \frac{n}{2})$ is said to be *in dimension* k .

Theorem 8 For all n even :

- if $2^{p+1} + 2 \leq n \leq 3 \times 2^p - 4$, then $G(n) \leq \frac{n \times (g_n - 2)}{2}$;
- if $3 \times 2^p - 2 \leq n \leq 2^{p+1} - 2$, then $G(n) \leq \frac{n \times (g_n - 1)}{2}$.

Proof : We recall first that in that case, $g_n = p + 2$. If we build the Knödel graph $W_{p,n}$ with $n \leq 3 \times 2^p - 4$, we can find a gossip scheme showing that this graph is a gossip graph. Thus the first upper bound for $G(n)$. If $n > 3 \times 2^p - 4$, then we show that $W_{p+1,n}$ is a gossip graph, which gives the second upper bound. We refer to [Fer97] for the detailed proof of this theorem.

6.2.2 Definitions

- Let's suppose we have two graphs G_1 and G_2 , both of order n . We call *composition of two graphs G_1 and G_2* any graph G of order $2n$ built by adding any perfect matching between G_1 and G_2 .
- A *self-composition* of a graph G is a composition of G by itself. For instance, the cartesian product by K_2 of any graph G is a self-composition of G .

6.2.3 Composition of two MGG_n

Theorem 9 For all n even, $G(2n) \leq 2 \times G(n) + n$.

Proof : If we self-compose a MGG_n , we get a graph with $2 \times G(n) + n$ edges, able to gossip in $g_n + 1 = g_{2n}$, as shown in Figure 2. We then get a gossip graph with $2 \times G(n) + n$ edges.

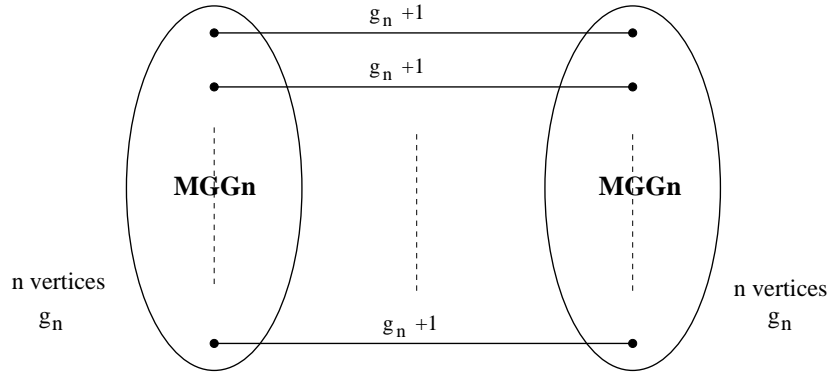


Figure 2: A gossip graph with $2 \times n$ vertices

Corollary 2 If a graph G is a MGG of order $2^p - 2$ ($p \geq 5$), then any self-composition of G is a MGG of order $2^{p+1} - 4$.

Proof : Labahn [Lab93] showed that $G(2^p - 2) = (p - 1) \times (2^{p-1} - 1)$ and that $G(2^{p+1} - 4) = p \times (2^p - 2)$. Then the self-composition of G is a gossip graph, following the scheme shown in Figure 2, with a minimum number of edges.

6.3 Gossip graphs with $2n_1 + n_2$ vertices (n_i even)

Theorem 10 For all (n_1, n_2) such that n_i even, $n_i \leq 2^{p-1}$ ($i \in \{1, 2\}$), $2 \times n_1 \geq n_2$ and $2 \times n_1 + n_2 > 2^p$, $G(2 \times n_1 + n_2) \leq 2 \times G(n_1) + G(n_2) + n_1 + \frac{n_2}{2}$.

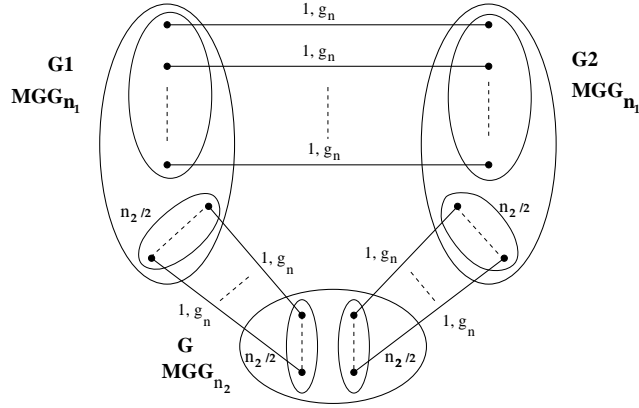


Figure 3: A gossip graph with $2 \times n_1 + n_2$ vertices

Proof : The constraints on n_1 and n_2 ensure us that g_n and g_{n_i} differ from 2 time units at least for $i \in \{1, 2\}$. Let's now take 2 MGG_{n_1} and a MGG_{n_2} . If we compose subgraphs of those MGG s as shown in Figure 3, and if we value the edges added for those compositions by 1 and g_n while the three MGG s gossip between 2 and $(g_n - 1)$, we get a gossip graph with $2 \times G(n_1) + G(n_2) + n_1 + \frac{n_2}{2}$ edges.

6.4 Gossip graphs with $2n + 1$ vertices

Theorem 11 For all $n \neq 2^p$, $G(2n + 1) \leq G(n) + G(n + 1) + n$.

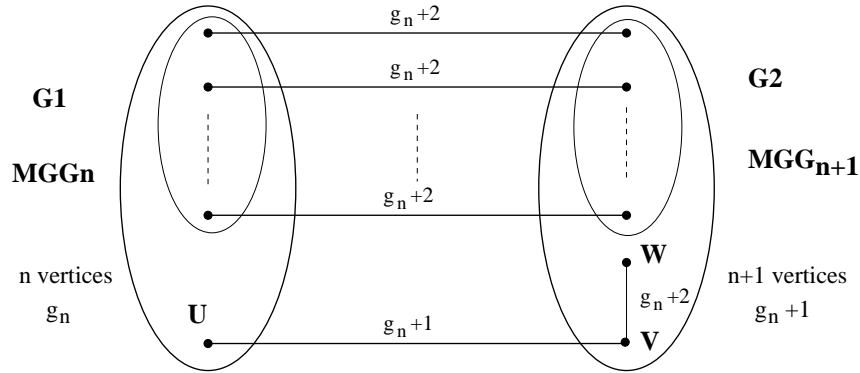


Figure 4: A gossip graph with $2 \times n + 1$ vertices

Proof : Apparently, there are two cases depending on the parity of n , but the proof is similar. We present here only the case n even.

If we suppose n even, we get $g_{2n+1} = g_{n+1} + 1 = g_n + 2$. Following the valuation scheme shown in Figure 4, we get a gossip graph. Let G_1 gossip in g_n time units and G_2 gossip in $(g_n + 1)$ time units. Moreover, $(n + 1)$ being odd, there's a vertex V in G_2 such that V is expert of G_2 in g_n time units. By valuating UV with $(g_n + 1)$, U and V become expert of the whole graph. By valuating an existing edge VW of G_2 with $(g_n + 2)$, W becomes expert too. The same goes for

the remaining vertices when composing $G_1 - \{U\}$ by $G_2 - \{V, W\}$ and valuating the edges added for the composition with $(g_n + 2)$.

6.5 (1,q)-type edges in a MGG_n

Theorem 12 *For all n odd such that $2^p + 1 \leq n \leq 2^{p+1} - 5$, we partition the possible values of n in four classes :*

- If $n = 4 \times m - 1$:
 - if $n \leq 3 \times 2^{p-1} - 9$, then $G(n) \leq m \times p - 1$;
 - if $n \geq 3 \times 2^{p-1} - 5$, then $G(n) \leq m \times (p + 1) - 1$.
- If $n = 4 \times m + 1$:
 - if $n \leq 3 \times 2^{p-1} - 11$, then $G(n) \leq (m + 1) \times p - 3$;
 - if $n \geq 3 \times 2^{p-1} - 7$, then $G(n) \leq (m + 1) \times (p + 1) - 3$.

Proof : We recall that, in that case $g_n = p + 2$. The aim is to obtain a gossip graph thanks to a construction based on Knödel graphs, to which we add some pendent edges. If $n = 4 \times m - 1$, take a Knödel graph of order $2m$, and add $(2m - 1)$ pendent edges to get a graph of order n . Otherwise, that is $n = 4 \times m + 1$, take a Knödel graph with $2(m + 1)$ vertices and add $(2m - 1)$ pendent edges. We know that there is a gossip in the Knödel graph in $p = g_n - 2$ time units (cf. Theorem 8). If we valuate the pendent edges with time units 1 and $(p + 2)$ while the Knödel graph gossips between time units 2 and $(p + 1)$, we see that this graph is a gossip graph. However, the number of edges of such a graph depends on the value of $2m$ - that is of n -, following Theorem 8. We refer to [Fer97] for a detailed proof of the theorem.

Theorem 13 *For all n even such that $2^{p-1} < n \leq 2^p$, and for all k odd such that $2^p - n < k < n$, $G(n + k) \leq G(n) + k$.*

Proof : Starting from G , a MGG_n , we add k (1,q)-type edges. Valuate those pendent edges with 1 and $(g_n + 1)$ while G gossips between 2 and g_n leads to a gossip graph with $G(n) + k$ edges.

Corollary 3 *For all n odd, $2^p + 1 \leq n \leq 2^{p+1} - 1$, $G(n) \leq n + (p - 2) \times 2^{p-1}$.*

Proof : As previously, we add $k = n - 2^p$ pendent edges to the hypercube Q_p , and valuate the edges the same way in order to get a gossip graph.

Theorem 14 *If G is a MGG_n with $2r$ vertices of degree 1 (n odd) such that $2^p + 2r + 1 \leq n \leq 2^{p+1} - 1$, then $G(n - 2r) \leq G(n) - 2r$.*

Proof : As, for these values of n , $g_{n-2r} = g_n$, we get a gossip graph with $G(n) - 2r$ edges and $(n - 2r)$ vertices, removing from the MGG_n $2r$ vertices of degree 1. We recall this removal can be done thanks to Theorem 1.

7 Lower bounds for G_n

7.1 Immediate property

Theorem 15 *For all $n \geq 4$, $G(n) \geq n$.*

Proof : As all $MGGs$ are connected, we know that $G(n) \geq n - 1$ for all n . Moreover, Property 4 yields $g_n \geq 2 \times \lceil \log_2 n \rceil - 1$ in a tree for all n . As $g_n = \lceil \log_2 n \rceil$ for n even and $g_n = \lceil \log_2 n \rceil + 1$ for n odd, we show that trees can't be $MGGs$ for any $n \geq 4$. Consequently, $G(n) \neq n - 1$.

7.2 Lower bounds for $G(2m)$

Theorem 16 For all n even such that $2^{p-1} + 2 \leq n \leq 3 \times 2^{p-2} - 2$ and $p \geq 4$, $G(n) \geq \lceil \frac{5n}{4} \rceil$.

Proof : The complete proof is in [Fer97]. As seen in Theorems 1 and 2, we know that there is no vertex of degree 1, and that if there is a vertex of degree 2, 1 and g_n valuate its incident edges. Moreover, Theorem 5 yields that there can't be two adjacent vertices of degree 2, except for $n = 2^p + 2$. We then show in Lemma 1 that if we have s vertices of degree at least 3, there will be at most s vertices of degree 2, and standard calculations yield the asserted formula.

Theorem 17 For all n even such that $2^p - 2^d + 2 \leq n \leq 2^p$ with $p \geq d + 2 \geq 5$, $G(n) \geq \lceil \frac{n(2p-2d+3)}{4} \rceil$.

Proof : The complete proof is in [Fer97]. Theorem 3 yields that for such values of n there is no vertex of degree less or equal to $(p-d)$. Hence, the minimum degree of any vertex will be $(p-d+1)$ in a MGG_n . We then show in Lemma 3 that if we have s vertices of degree $(p-d+1)$ in a MGG_n , there are at least s vertices of degree $(p-d+2)$ or more. Standard calculations then yield the asserted formula.

Theorem 18 For all n even such that $2^p - 2^d \leq n \leq 2^p$ with $p \geq d + 2$, $G(n) \geq \frac{(p-d+1) \times n}{2}$.

Remark : Note that the lower bound for $G(n)$ given in the previous theorem (Theorem 17) is better than this one. However, this bound is available for $n = 2^p - 2^d$, which is not the case in Theorem 17.

Proof : As seen in Theorem 3, there cannot be any vertex of degree less or equal to $(p-d)$ in a MGG_n for such values of n ; hence the result.

Corollary 4 For all (p, q) such that $q \geq 2$ and $p \geq q+2$, $(p-q+1) \times (2^{p-1} - 2^{q-1}) \leq G(2^p - 2^q) \leq (p-1) \times (2^{p-1} - 2^{q-1})$.

Proof : This is done by induction on q . First, let us show the result for $q = 2$ and any $p \geq 4$. Suppose $q = 2$; then, for $p = 4$, that is $n = 12$, we know that $G(12) = 18$ [Lab93], and the result follows the formula. Similarly, for $p = 5$, that is $n = 28$, the result, which also follows the asserted formula, is shown in Theorem 25. Finally, for any $p \geq 6$, Labahn [Lab93] showed that $G(2^p - 4) = (p-1) \times (2^{p-1} - 2)$.

Now suppose the result is true for q and any $p \geq q+2$, and let us prove it is true for $q+1$ and any $p \geq (q+1)+2$. First, as $n = 2^p - 2^{q+1}$, Theorem 3 yields that there cannot be any vertex of degree less or equal to $p-q-1$. Hence Theorem 18 yields $G(2^p - 2^{q+1}) \geq (p-q) \times (2^{p-1} - 2^q)$. Moreover, Theorem 9 gives us an upper bound for $G(2m)$ with m even. Let $m = 2^{p-1} - 2^q$. We know that $G(m) \leq (p-2) \times (2^{p-2} - 2^{q-1})$, since $p-1 \geq q+2$. As Theorem 9 yields $G(2m) \leq 2 \times G(m) + m$, standard calculations give us the needed upper bound for $G(2^p - 2^{q+1})$.

7.3 Lower bounds for $G(2m+1)$

Theorem 19 For all n odd, $n \geq 13$, $G(n) \geq n + 1$.

Proof : Theorem 15 shows that $G(n) \geq n$ for all $n \geq 4$. In the case $G(n) = n$, it is easy to see that there can only be vertices of degree 2 and/or vertices contributing to $(1, 3)$ -edges. In that case, whatever the number of $(1, 3)$ -type edges is, we always can find a MBT containing less than n vertices for any $n \geq 13$. Which yields $G(n) \neq n$.

Theorem 20 For all n odd such that $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, $G(n) \geq \lceil \frac{7n}{6} \rceil$.

Proof : As for Theorem 16, the complete proof can be found in [Fer97]. Theorem 4 yields that there is no $(1, 3)$ -type edge. Theorems 2 and 6 respectively say that if there is a vertex of degree 2, 1 and g_n valuate its incident edges and that a vertex of degree 2 has at most one neighbour

of degree 2. So that there is a limited number of vertices of degree 2. Actually, we can show that if we have s vertices of degree at least 3 (vertices incident to a $(1, q)$ -type edge excluded) in a MGG_n , then there are at most $2s$ vertices of degree 2. Standard calculations then yield the asserted formula.

Theorem 21 For all $n = 2^p - 1$, $G(n) \geq \lceil \frac{6n}{5} \rceil$.

Proof : In that case, we get some more constraints than previously. Consequently, we show that if there are s vertices of degree at least 3 (vertices incident to $(1, q)$ -type edges excluded) in a MGG_n , then there are at most $\frac{3s}{2}$ vertices of degree 2 ([Fer97]). This provides immediately the asserted formula.

8 Some particular cases

8.1 Tights bounds for $G(13)$

Theorem 22 $16 \leq G(13) \leq 17$.

Proof : Corollary 3 and Theorem 20 give us respectively the lower and upper bound for $G(13)$.

Corollary 5 If there are at least six $(1, 4)$ -type edges in a MGG_{13} , then $G(13) = 17$.

Proof : Proof of Theorem 20 yields that for $n = 13$, $G(n) \geq \frac{7n+r}{6}$, where r is the number of $(1, q)$ -type edges. Consequently, if $r \geq 6$ then $G(13) \geq 17$. Theorem 22 then yields $G(13) = 17$. Moreover, the only $(1, q)$ -type edges that can exist are $(1, 4)$ -type ones because :

- $(1, 3)$ -type edges cannot exist for $n = 13$ (cf. Theorem 4), therefore $q \geq 4$;
- As $g_{13} = 5$ and a $(1, q)$ -type edge is valuated by two time units (namely 1 and g_n , cf. Theorem 1), $q \leq 4$.

8.2 Determination of $G(15)$

Theorem 23 $G(15) = 19$.

Proof : Theorem 21 and Corollary 3 give us respectively the lower and upper bound : $18 \leq G(15) \leq 19$. Now let's suppose $G(15) = 18$. As seen in proof of Theorem 21, in that case we have 6 vertices of degree 3 and 9 vertices of degree 2. But, thanks to tools provided by the proof of Theorem 21, we show that in that case no MGG_{15} can be built with such vertices. Hence $G(15) \neq 18$. Thanks to Corollary 3, we know that the hypercube H_3 to which 7 pendent edges are added is a MGG on 15 vertices.

8.3 Determination of $G(24)$

Theorem 24 $G(24) = 36$.

Proof : As seen in Theorem 18, we know that $G(24) \geq \frac{3 \times 24}{2}$, that is $G(24) \geq 36$. Moreover, in their study of Minimum Linear Gossip Graphs, Fraigniaud and Peters [FP94] gave a Minimum Linear Gossip Graph with 24 vertices and $G_{1,1}(24) = 36$ edges, where $G_{\beta,\tau}$ is the minimum number of edges of a graph that achieves gossiping in $g_{\beta,\tau}(n) = \lceil \log_2 n \rceil \beta + (n-1)\tau$ time units. As $G_{1,1}(n) \geq G(n)$ for any n , $G(24) \leq 36$, and therefore $G(24) = 36$. We refer to [FP94] for an example of MGG_{24} .

8.4 Determination of $G(28)$

Theorem 25 $G(28) = 56$.

Proof: First notice that Labahn [Lab93] gave the value of $G(n)$ for $n = 2^p - 4$, but that this only holds for $p \geq 6$. We know that $G(28) \geq 56$, thanks to Theorem 3, where $d = 4$. Moreover, the construction given in Theorem 9 to get an upper bound for $G(2n)$ yields $G(2n) \leq 2 \times G(n) + n$. As $G(14) = 21$ [Lab93], we get $G(28) \leq 56$. Hence $G(28) = 56$, and the self-composition of a MGG_{14} is a MGG on 28 vertices.

8.5 Bounds for $G(n)$ for $1 \leq n \leq 32$

To the author's knowledge, no bounds on $G(n)$ are available in the literature when $G(n)$ is not precisely known. Therefore, for $n = 13$, $n = 15$ and $17 \leq n \leq 32$, the results shown below are believed to be the best known so far. Moreover, the results printed in bold characters indicate new exact results for $G(n)$, while the asterisk indicates optimality.

These bounds are given thanks to the different theorems and corollaries presented above, except for $G(20) \geq 26$ and $G(22) \geq 31$, which are deduced from [MS94] (where we have $B(20) = 26$ and $B(22) = 31$) and Property 1.

n	Lower	Upper	n	Lower	Upper	n	Lower	Upper	n	Lower	Upper
1	0	0*	9	9	9*	17	18	20	25	30	32
2	1	1*	10	13	13*	18	23	27	26	46	52
3	2	2*	11	11	11*	19	20	22	27	32	34
4	4	4*	12	18	18*	20	26	30	28	56	56*
5	5	5*	13	16	17	21	22	27	29	34	45
6	6	6*	14	21	21*	22	31	41	30	60	60*
7	7	7*	15	19	19*	23	24	29	31	38	47
8	12	12*	16	32	32*	24	36	36*	32	80	80*

Figure 5: Bounds for $G(n)$ ($1 \leq n \leq 32$)

8.6 Sum-up of the results for n even

8.6.1 Lower bounds for $G(n)$

n	$2^{p-1} + 2 \dots 3 \times 2^{p-2} - 2$	$3 \times 2^{p-2}$	$3 \times 2^{p-2} + 2 \dots 2^p - 2^d - 2$	$2^p - 2d$	$2^p - 2^d + 2 \dots 2^p$
$G(n)$	$\lceil \frac{5n}{4} \rceil$	$\frac{3n}{2}$	$\lceil \frac{7n}{4} \rceil$	$\frac{n(p-d+1)}{2}$	$\lceil \frac{n(2p-2d+3)}{4} \rceil$

8.6.2 Upper bounds for $G(n)$

n	$2^{p-1} + 2 \dots 3 \times 2^{p-2} - 4$	$3 \times 2^{p-2} - 2 \dots 2^p - 2$	2^p
$G(n)$	$\frac{n(p-2)}{2}$	$\frac{n(p-1)}{2}$	$\frac{np}{2}$

8.7 Sum-up of the results for n odd

8.7.1 Lower bounds for $G(n)$

n	$2^{p-1} + 1 \dots 3 \times 2^{p-2} - 1$	$3 \times 2^{p-2} + 1 \dots 2^p - 3$	$2^p - 1$
$G(n)$	$n + 1$	$\lceil \frac{7n}{6} \rceil$	$\lceil \frac{6n}{5} \rceil$

8.7.2 Upper bounds for $G(n)$

n	$n = 4 \times m - 1$		$n = 4 \times m + 1$	
	$2^{p-1} + 1 \dots 3 \times 2^{p-2} - 9$	$3 \times 2^{p-2} - 5 \dots 2^p - 1$	$2^{p-1} + 1 \dots 3 \times 2^{p-2} - 11$	$3 \times 2^{p-2} - 7 \dots 2^p - 1$
$G(n)$	$m(p-1) - 1$	$mp - 1$	$(m+1)(p-1) - 3$	$(m+1)p - 3$

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9 Annex : complete proof of theorems

9.1 Proof of Theorem 1, page 4

We are going to give here the complete proof of Theorem 1. Let's suppose we have a cut-edge AB dividing a MGG_n in G_1 and G_2 , as shown in Figure 6. Let's suppose too that $AB = \{t_1 \dots t_p\}$, where $1 \leq t_1 < t_2 \dots < t_p \leq g_n$.

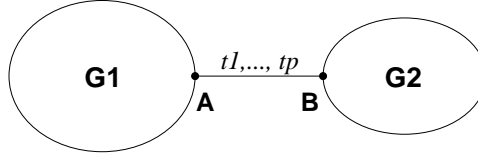


Figure 6: A cut-edge AB in a MGG_n

Let t_i be the smallest valuation on AB such that A knows the information of each vertex of G_1 in time $(t_i - 1)$ at most. Analogously, let t_j be the smallest valuation on AB such that B knows all the information of G_2 in time $(t_j - 1)$ at most. We then get the four following inequalities :

- $t_i - 1 \geq \lceil \log_2(|V(G_1)|) \rceil$, as there has to be a broadcast from G_1 to A in at most $(t_i - 1)$ time units. Yet, we know that the minimum time to broadcast in the graph G_1 is $\lceil \log_2(|V(G_1)|) \rceil$.
- Analogously, $t_j - 1 \geq \lceil \log_2(|V(G_2)|) \rceil$, considering that there must be a broadcast from G_2 to B in $(t_j - 1)$ time units at most.
- $g_n - t_i \geq \lceil \log_2(|V(G_2)|) \rceil$, because after time unit t_i , B knows all the information of G_1 and needs to broadcast it through G_2 between time units $(t_i + 1)$ and g_n .
- $g_n - t_j \geq \lceil \log_2(|V(G_1)|) \rceil$. Analogously, if we consider the vertex A , it has $(g_n - t_j)$ time units to transmit the information through G_1 .

If we add for instance the first inequality to the third, we get :

$$g_n - 1 \geq \lceil \log_2 a \rceil + \lceil \log_2 b \rceil \quad \text{(I1)}$$

where $a = |V(G_1)|$ and $b = |V(G_2)|$, and $a + b = n$. W.l.o.g., let us consider that $a \geq \lceil \frac{n}{2} \rceil$.

Now suppose n even, and let us distinguish two cases :

- $b = 1$. In that case, $a = n - 1$ and (I1) becomes $\lceil \log_2 n \rceil \geq \lceil \log_2(n - 1) \rceil + 1$, which is false for any $n \geq 4$.
- $2 \leq b \leq \frac{n}{2}$. In that case, (I1) becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 n \rceil + 1$, as we supposed $a \geq \frac{n}{2}$. However, this is false for any n .

Hence, if n is even ($n \geq 4$), there is no cut-edge in a MGG .

Let n be odd, and a and b as above. Let us distinguish three cases ;

- $b = 1$, that is BA is a pendent edge. Then (I1) becomes $\lceil \log_2 n \rceil \geq \lceil \log_2(n - 1) \rceil$, which is true for any n . In that case, the four inequalities above lead to two cases :
 1. If $n \neq 2^p + 1$, we get $t_j \leq 1$ and $t_i \geq g_n$;
 2. If $n = 2^p + 1$, we get $t_j \leq 2$ and $t_i \geq g_n - 1$, which leads to four possible valuation schemes for the pendent edge BA . Let us show that only the case $\{t_j, t_i\} = \{1, g_n\}$ is possible.

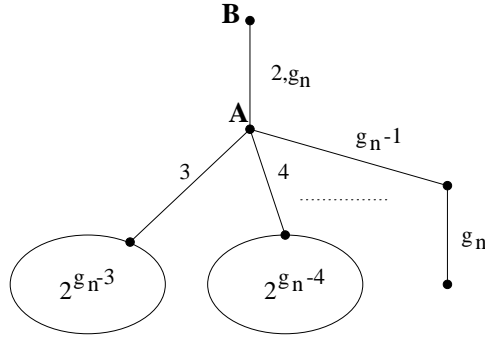


Figure 7: *MBT* rooted in B with $BA = \{2, g_n\}$

- If $\{t_j, t_i\} = \{2, g_n\}$, the *MBT* rooted in B , as shown in Figure 7, holds $|V(MBT)| = 1 + 1 + 2^{g_n-3} + 2^{g_n-4} + \dots + 2$ vertices, that is $|V(MBT)| = 2^{g_n-2}$. As $n = 2^{g_n-2} + 1$, we show that this case is impossible.
- If $\{t_j, t_i\} = \{1, g_n - 1\}$, reasoning symmetrically with the *RBT* rooted in B , as shown in Figure 8, we get the same result, e.g. $|V(RBT)| = 2^{g_n-2}$.

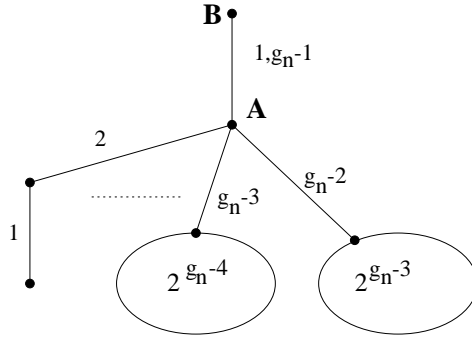


Figure 8: *RBT* rooted in B with $BA = \{1, g_n - 1\}$

- If $\{t_j, t_i\} = \{2, g_n - 1\}$, then we show in Figure 9 that the *MBT* rooted in B holds $|V(MBT)| = 2^{g_n-2} - 1$ vertices, which is strictly less than needed.

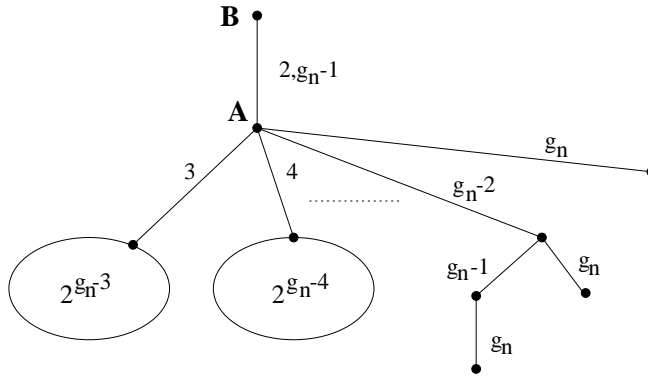


Figure 9: *MBT* rooted in B with $BA = \{2, g_n - 1\}$

- Finally, if $\{t_j, t_i\} = \{1, g_n\}$, the *MBT* as well as the *RBT* rooted in B both hold $|V(MBT)| = 2^{g_n-1} - 1$ vertices, which remains possible.

Hence the *MBTs* and *RBTs* rooted in B hold enough vertices only in the case $BA = \{1, g_n\}$.

- $b = 2$. In that case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2(n-2) \rceil + 1$, which is true only for $n = 2^p + 1$. Let $V(G_1) = \{B, B_1\}$. Then BB_1 is a pendent edge and, as seen above, we have $BB_1 = \{1, g_n\}$. Moreover, the first four inequalities yield $t_j = 2$ and $t_i = g_n - 1$. If we consider then the *MBT* rooted in B_1 , as shown in Figure 10, we get $|V(MBT)| = 2^{g_n-2}$ for any $g_n \geq 4$, that is for any $n \geq 5$. As $n = 2^{g_n-2} + 1$, we show that this case is impossible.

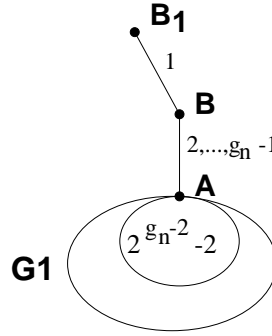


Figure 10: *MBT* rooted in B_1 where $a = 2^{g_n-2} - 1$ and $b = 2$

- $3 \leq b \leq \frac{n-1}{2}$. In that case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2(n+1) \rceil + 1$, as we supposed $a \geq \frac{n+1}{2}$. However, this is false for any n .

Finally, if there is a cut-edge AB in a MGG_n where n is odd ($n \geq 5$), then it is a pendent edge ; moreover, $AB = \{1, g_n\}$.

9.2 Proof of Theorem 2, page 5

If 1 doesn't valuate AB nor AC , we get the *MBT* rooted in A shown in Figure 11, which holds $|V(MBT)| = 3 \times 2^{g_n-3} + 1$ vertices.

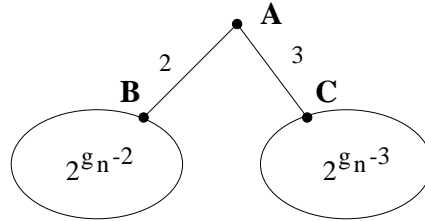


Figure 11: *MBT* rooted in A ($1 \notin AB, 1 \notin AC$)

Consequently, the only possible case is when $n = 3 \times 2^{g_n-3} + 1$. In that case, as $|V(MBT)|$ has exactly n vertices, we need to have, w.l.o.g., $2 \in AB$ and $3 \in AC$. Then suppose $(g_n - t_1)$ and $(g_n - t_2)$ are the maximum values respectively on AB and AC , with $t_1 \neq t_2$ and $t_i \geq 0$ ($i \in \{1, 2\}$). We get the *MBT* rooted in A shown in Figure 12.

This *MBT* holds :

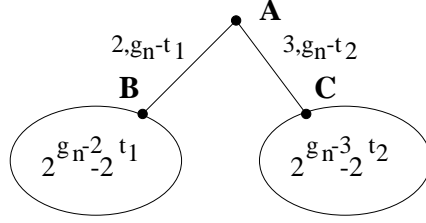


Figure 12: *MBT* rooted in A

- $|V(MBT)| = 3 \times 2^{g_n - 3} + 1$ if $g_n - t_1 = 2$ and $g_n - t_2 = 3$;
- $|V(MBT)| = 3 \times 2^{g_n - 3} + 1 - 2^{t_2}$ if $g_n - t_1 = 2$ and $g_n - t_2 \neq 3$;
- $|V(MBT)| = 3 \times 2^{g_n - 3} + 1 - 2^{t_1}$ if $g_n - t_2 = 3$ and $g_n - t_1 \neq 2$;
- $|V(MBT)| = 3 \times 2^{g_n - 3} - 2^{t_1} - 2^{t_2} + 1$ otherwise.

Which shows the *MBT* rooted in A holds enough vertices only in the first case. However, in this case we get a *RBT* rooted in A with 7 vertices, as $(g_n - t_1)$ is the biggest valuation on AB and $(g_n - t_2)$ the biggest valuation on AC . As we supposed $n \geq 9$, 1 valuates necessarily AB or AC .

By a completely symmetrical proof (looking at *RBTs* instead of *MBTs*, and vice-versa), we prove that g_n also valuates either AB or AC .

9.3 Proof of Theorem 3, page 5

Let A be a vertex of degree e , where $1 \leq e \leq d$. Then the *MBT* rooted in A , as shown in Figure 13, holds $|V(MBT)| = (2^e - 1) \times 2^{g_n - e} + 1$ vertices.

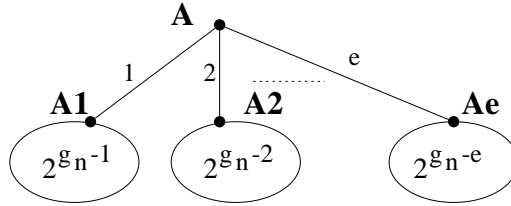


Figure 13: *MBT* rooted in A of degree e

Clearly, this is only possible when $e = d$ and $n = (2^d - 1) \times 2^{p-d}$. In that case, let's suppose that one of the edges AA_i ($1 \leq i \leq d$) is such that $i \notin AA_i$. The *MBT* rooted in A , as shown in Figure 14, then holds $|V(MBT)| = 2^{g_n} - 2^{g_n - d - 1} - 2^{g_n - i} + 1$ vertices, which is strictly less than n for all i since $i \leq d \leq g_n - 2$.

Consequently, $i \in AA_i$ for any $i \in \{1, 2, \dots, d\}$. Symmetrically, thanks to the *RBT* rooted in A , we show that the valuations $g_n, g_n - 1, \dots, g_n - d + 1$ valuate necessarily the edges incident to A . As $g_n \geq d + 2$, time units g_n and $(g_n - 1)$, at least, are distinct from $1, 2, \dots, d$. Hence, the number of vertices of the *MBT* rooted in A will remain the same wherever the valuations g_n and $(g_n - 1)$ may be placed. Consequently, suppose w.l.o.g. that $g_n \in AA_1$ and $(g_n - 1) \in AA_2$. Then the *MBT* rooted in A , as shown in Figure 15, holds at most $(2^d - 1) \times 2^{g_n - d} - 2$ vertices, which is strictly less than n .

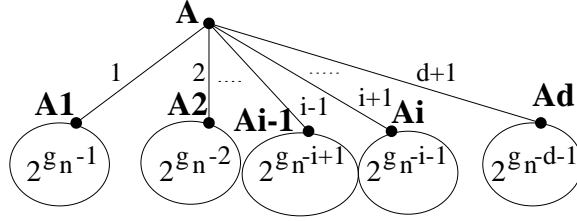


Figure 14: MBT rooted in A where $i \notin AA_i$

Consequently, any MGG_n for these values of n cannot hold any vertex of degree less or equal to d .

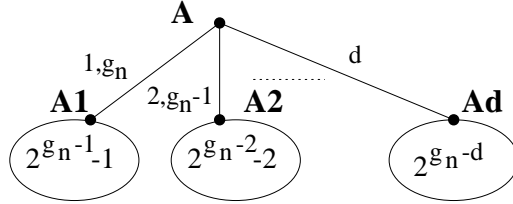


Figure 15: MBT rooted in A

9.4 Proof of Theorem 4, page 6

Suppose AB is a $(1, 3)$ -type edge. We already know that $AB = \{1, g_n\}$ thanks to Theorem 1. We recall that we supposed $n \geq 13$, which means $g_n \geq 5$. The MBT rooted in A , as shown in Figure 16, has $|V(MBT)| = 3 \times 2^{g_n-3} + 2$ vertices. Because n is odd, the only possible case is when $n = 3 \times 2^{g_n-3} + 1$.

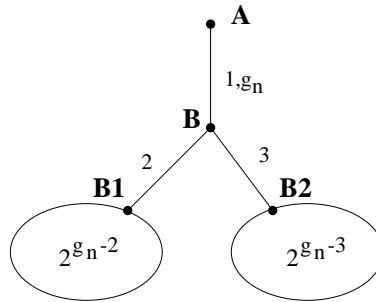


Figure 16: MBT rooted in A where AB is a $(1,3)$ -type edge

In that case, let's suppose that $2 \notin BB_1 \cup BB_2$. The MBT rooted in A , as shown in Figure 17, has $|V(MBT)| = 3 \times 2^{g_n-4} + 2$ vertices, which is strictly less than n for any $g_n \geq 4$.

Analogously, suppose $3 \notin BB_1 \cup BB_2$. Figure 18 yields that the MBT rooted in A holds $|V(MBT)| = 5 \times 2^{g_n-4} + 2$, which is also strictly less than n for any $g_n \geq 5$.

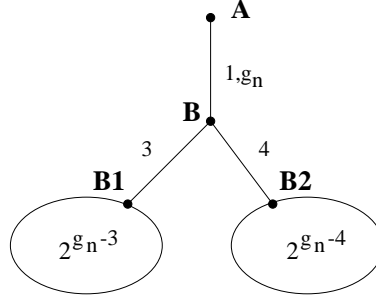


Figure 17: *MBT* rooted in A supposing $2 \notin BB_i$

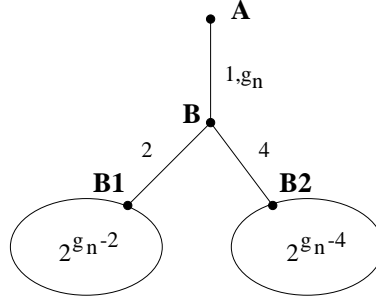


Figure 18: *MBT* rooted in A supposing $3 \notin BB_i$

Symmetrically, using the *RBTs* rooted in A in each case, we prove that $(g_n - 1) \in BB_j$ and $(g_n - 2) \in BB_k$. We then get two possible valuation schemes, as shown in Figure 19.

We know that $g_n \geq 5$. Let us distinguish two cases :

- If $g_n = 5$, then the only possible valuation scheme is the left one in Figure 19 and $|V(MBT)| = 3 \times 2^{g_n-3}$.
- If $g_n > 5$, then the two valuation schemes lead to a *MBT* rooted in A with $|V(MBT)| = 3 \times 2^{g_n-3} - 4$ vertices.

In every case, we see that none of the *MBTs* rooted in A holds enough vertices : for those values of n , there is no (1,3)-type edge.

9.5 Proof of Theorem 5, page 6

Suppose we have two vertices of degree 2 adjacent, A and B , and A_1 (resp. B_1) is the other vertex adjacent to A (resp. B). Knowing that 1 and g_n value the edges adjacent to A and B , we get four possibilities. Let us distinguish them :

1. Suppose $1 \in AB$. Let us denote by p (resp. q) the smallest valuation on AA_1 (resp. BB_1). As shown in Figure 20, the *MBT* rooted in A has $|V(MBT)| = 2^{g_n-p} + 2^{g_n-q} + 2$ vertices. We notice that $\forall (p, q)$, $|V(MBT)| \leq 2^{g_n-1} + 2$, which means that the only possible case would be for $n = 2^{g_n-1} + 2$, and, in that case, $p = q = 2$.

Now, considering where the valuation g_n could be, we get two cases.

- (a) If $g_n \in AB$, then, reasoning symmetrically, we can prove that $(g_n - 1) \in AA_1$ and $(g_n - 1) \in BB_1$. Unless $g_n = 2$, the *MBT* rooted in A , as shown in Figure 21, has

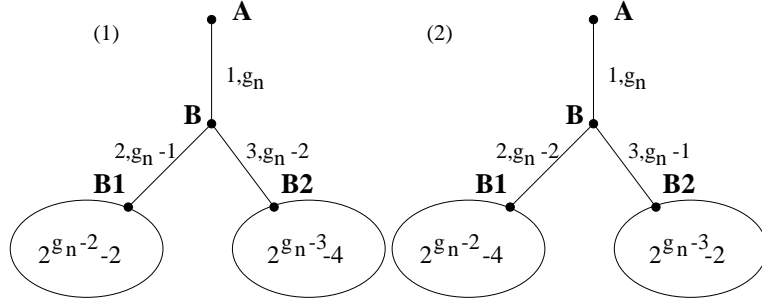


Figure 19: *MBT*s rooted in A where AB is a $(1,3)$ -type edge

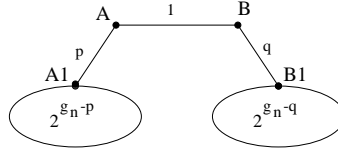


Figure 20: *MBT* rooted in A

$|V(MBT)| = 2^{g_n-1} - 2$ vertices, which is strictly less than n . If $g_n = 2$, however, we get to $n = 6$. As we supposed $n \geq 8$, we don't have to consider this case.

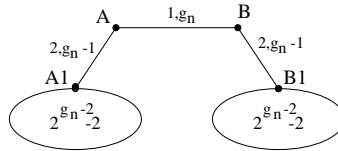


Figure 21: *MBT* rooted in A

- (b) If $g_n \in AA_1$ and $g_n \in BB_1$, we get the valuation scheme shown in Figure 22. For any $g_n \geq 3$, this leads to a *RBT* rooted in A with $|V(MBT)| = 2^{g_n-1}$ vertices.

As a consequence, if two vertices A and B of degree 2 are adjacent in a MGG_n when n is even, then $1 \notin AB$.

2. If $1 \in AA_1$ and $1 \in BB_1$, let's suppose $|AA_1| = 1$, as shown in Figure 23. This leads to a *MBT* rooted in A with $|V(MBT)| = 2^{g_n-2} + 3$ vertices, which is strictly less than n for any $g_n \geq 3$; then, $|AA_1| \neq 1$. Symmetrically, we conclude that $|BB_1| \neq 1$ too.

Now let's consider the two different cases :

- (a) If $g_n \in AB$, then we know (thanks to the first part of the proof) that $(g_n-1) \in AA_1$ and $(g_n-1) \in BB_1$. Figure 24 then leads to a *RBT* rooted in A with $|V(RBT)| = 2^{g_n-1}$ vertices, which is strictly less than n .
- (b) If $g_n \in AA_1$ and $g_n \in BB_1$, let p (resp. q) be the smallest valuation on AA_1 (resp. BB_1) such that $p \neq 1$ (resp. $q \neq 1$). Consequently, $p \geq 3$ and $q \geq 3$. Then, the *MBT*s rooted in A (as shown in Figure 25) and B respectively hold :

- $|V(MBT_A)| = 2^{g_n-1} - 2^{g_n-p} + 2^{g_n-q} + 2$ vertices ;

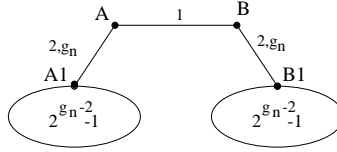


Figure 22: *MBT* rooted in A

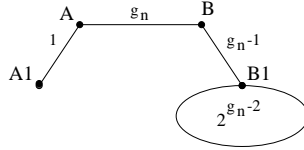


Figure 23: *RBT* rooted in A , supposing $|AA_1| = 1$

- $|V(MBT_B)| = 2^{g_n-1} + 2^{g_n-p} - 2^{g_n-q} + 2$ vertices ;

Consequently, n can at most be equal to $2^{g_n-1} + 2$, and in that case $p = q$. Following the hypothesis that $2^{g_n-1} + 2 \leq n \leq 2^{g_n}$, then $n = 2^{g_n-1} + 2$. Let's distinguish two cases :

- If $p \neq g_n$, then the *MBT* rooted in A , as shown in Figure 26, holds $|V(MBT)| = 2^{g_n-1}$ vertices, which is of course not enough ;
- If $p = q = g_n$, that is $AA_1 = \{1, g_n\}$ and $BB_1 = \{1, g_n\}$, then the *RBTs* and *MBTs* rooted in A and B all hold $|V(MBT)| = n$ vertices, which remains possible.

Then there cannot be any two vertices of degree 2 adjacent for any n even with $n \geq 8$ and $n \neq 2^{g_n-1} + 2$; if $n = 2^{g_n-1} + 2$, then necessarily $AA_1 = \{1, g_n\}$ and $BB_1 = \{1, g_n\}$.

9.6 Proof of Theorem 6, page 6

Let's suppose we have three vertices of degree 2, A, B, C such that the neighborhood of A is $\{B, C\}$. We know, thanks to Theorem 2 that 1 and g_n valuate the edges adjacent to these vertices. W.l.o.g., thanks to the symmetry in A , let's suppose $1 \in AB$. Then the *MBT* rooted in A , as shown in Figure 27 holds $|V(MBT)| = 3 \times 2^{g_n-3} + 3$ vertices, which remains possible for $n = 3 \times 2^{g_n-3} + 3$ and $n = 3 \times 2^{g_n-3} + 1$.

In both cases, let's show that 2 valuates AC . If $2 \notin AC$, then, as shown in Figure 28, the *MBT* rooted in A has $|V(MBT)| = 5 \times 2^{g_n-4} + 3$ vertices, and $|V(MBT)| < n$ for any $g_n \geq 6$.

If $g_n = 5$, however, we need to introduce the valuation g_n to get the result. If $g_n \in AB$, then Theorem 2 implies that $g_n \in CC_1$, as shown in Figure 29 (left figure). Analogously, if $g_n \in AC$ then $g_n \in BB_1$ too. We recall that in that case $g_n = 5$ and that we suppose $n \in \{13, 15\}$. Figure 29 shows that, wherever the valuation g_n is, we get a *MBT* rooted in A with 12 vertices.

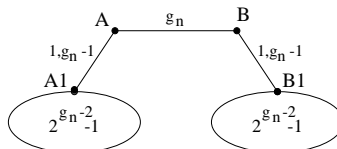


Figure 24: *RBT* rooted in A

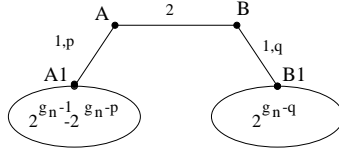


Figure 25: *MBT* rooted in A

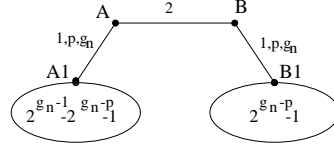


Figure 26: *MBT* rooted in A where $p \neq g_n$

Then, we know that $1 \in AB$ and $2 \in AC$. By a completely analogous proof, we show that, similarly, $2 \in BB_1$ and $3 \in CC_1$. Moreover, by a symmetrical proof (looking at *RBTs* instead of *MBTs*, and vice-versa), we show that g_n is either in AB or AC , and that :

- If $g_n \in AB$, $(g_n - 1) \in AC$, $(g_n - 1) \in BB_1$ and $(g_n - 2) \in CC_1$;
- If $g_n \in AC$, $(g_n - 1) \in AB$, $(g_n - 1) \in CC_1$ and $(g_n - 2) \in BB_1$.

Moreover, Theorem 2 implies that, since $1 \in AB$, then $1 \in CC_1$, and that :

- If $g_n \in AB$, then $g_n \in CC_1$;
- If $g_n \in AC$, then $g_n \in BB_1$.

The case $g_n \in AB$ refers to the left figure in Figure 30, as the case $g_n \in AC$ refers to the right one. In both cases, we show that the *MBT* rooted in A holds :

- $|V(MBT)| \leq 3 \times 2^{g_n-3}$ vertices for $g_n = 5$;
- $|V(MBT)| = 3 \times 2^{g_n-3} - 4$ vertices for $g_n > 5$.

We then prove that, in any case, there can't be a vertex of degree 2 adjacent to two vertices of degree 2 for any n odd such that $n \geq 3 \times 2^{g_n-3} + 1$ with $g_n \geq 5$.

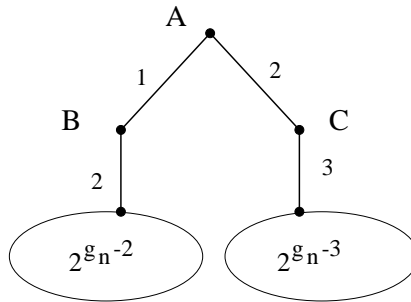


Figure 27: 3 adjacent vertices of degree 2 : *MBT* rooted in A

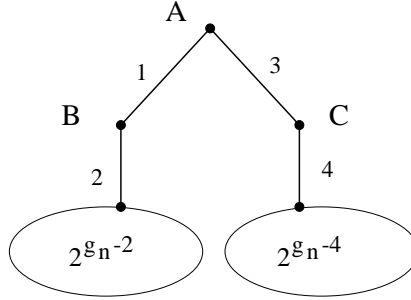


Figure 28: *MBT* rooted in A , supposing $2 \notin AB$

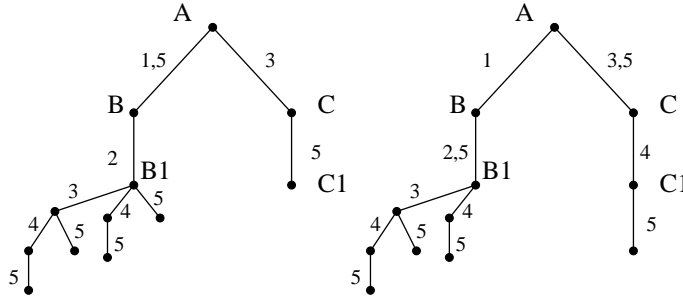


Figure 29: *MBTs* rooted in A where $g_n = 5$

9.7 Proof of Theorem 8, page 7

The aim is to show that :

- For n even such that $2^{p+1} + 2 \leq n \leq 3 \times 2^p - 4$, $W_{p,n}$ is a gossip graph ;
- For n even such that $3 \times 2^p - 2 \leq n \leq 2^{p+2} - 2$, $W_{p+1,n}$ is a gossip graph.

9.7.1 $2^{p+1} \leq n \leq 3 \times 2^p - 4$

Let us prove that $W_{p,n}$ is a gossip graph. First, we recall that $g_n = p + 2$. Let the gossip scheme be the following :

- For i in $1 \dots p$, vertices communicate through dimension $(i - 1)$ during time unit i ;
- Vertices communicate through dimension 0 during time unit $p + 1$;
- Vertices communicate through dimension $(p - 1)$ during time unit $p + 2$.

Now let us show that this gives a valid gossip scheme. It is easy to see that after a round $q \leq p$, a vertex $(1, j_0)$ knows the pieces of information of all the vertices (i, j) , $i = 1, 2$ and $j = j_0, j_0 + 1, \dots, j_0 + 2^{q-1} - 1 \pmod{\frac{n}{2}}$. Similarly, a vertex $(2, j_0)$ knows the pieces of information of all the vertices (i, j) , $i = 1, 2$ and $j = j_0, j_0 - 1, \dots, j_0 - 2^{q-1} + 1 \pmod{\frac{n}{2}}$ [FP94].

As every vertex communicates through dimension 0 during time unit $(p + 1)$, a vertex (i_0, j_0) knows all the pieces of information of vertices (i, j) , $i = 1, 2$ and $j = j_0 - 2^{p-1} + 1 \pmod{\frac{n}{2}}, \dots, j_0 - 1, j_0, j_0 + 1, \dots, j_0 + 2^{p-1} - 1 \pmod{\frac{n}{2}}$. Finally, there will be a communication through dimension $(p - 1)$. Let us consider a vertex $(1, j_0)$ and a vertex $(2, j_0 + 2^{p-1} - 1 \pmod{\frac{n}{2}})$. As seen above, $(1, j_0)$ knows the pieces of information of (i, j) , $i = 1, 2$ and $j = j_0 - 2^{p-1} + 1 \pmod{\frac{n}{2}}, \dots, j_0 -$

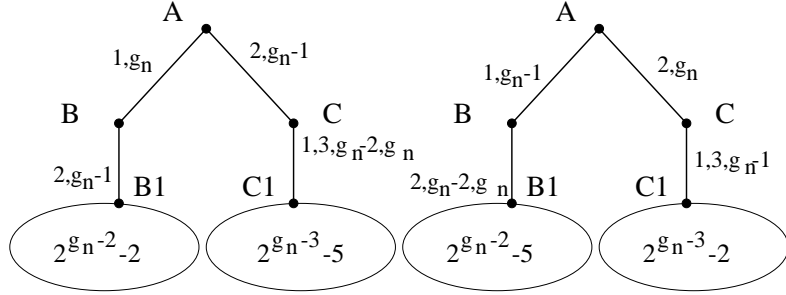


Figure 30: *MBTs* rooted in *A*

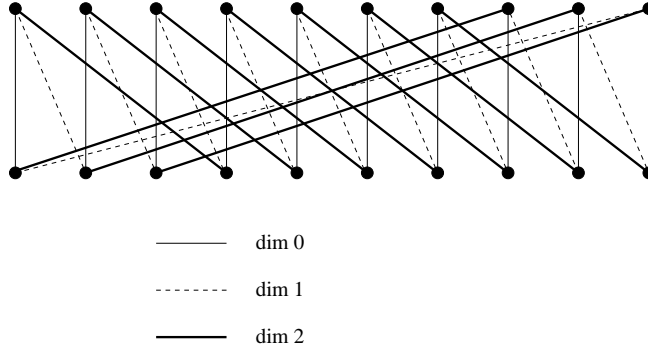


Figure 31: $W_{3,20}$ is a gossip graph

$1, j_0, j_0 + 1, \dots, j_0 + 2^{p-1} - 1 \pmod{\frac{n}{2}}$, and $(2, j_0 + 2^{p-1} - 1 \pmod{\frac{n}{2}})$ knows the pieces of information of (i, j) , $i = 1, 2$ and $j = j_0, j_0 + 1, \dots, j_0 + 2^p - 2 \pmod{\frac{n}{2}}$. During time unit $p + 2$, those two vertices communicate. Thus, they will know the pieces of information of all the other vertices iff $j_0 + 2^p - 2 \pmod{\frac{n}{2}} + 1 \geq j_0 - 2^{p-1} + 1 \pmod{\frac{n}{2}}$, that is $n \leq 3 \times 2^p - 4$.

Consequently, if $2^{p+1} + 2 \leq n \leq 3 \times 2^p - 4$, $W_{p,n}$ is a gossip graph and thus $G(n) \leq \frac{np}{2}$ where $p = g_n - 2$.

9.7.2 $3 \times 2^p - 2 \leq n \leq 2^{p+2} - 2$

Let us prove that $W_{p+1,n}$ is a gossip graph. Let the gossip scheme be the following :

- For i in $1 \dots p + 1$, vertices communicate through dimension $(i - 1)$ during time unit i ;
- Vertices communicate through dimension 0 during time unit $p + 2$;

Similarly to the previous case, it is easy to see that after the $(p + 1)$ -th round, a vertex $(1, j_0)$ knows all the pieces of information of vertices (i, j) , $i = 1, 2$ and $j = j_0, j_0 + 1, \dots, j_0 + 2^p - 1 \pmod{\frac{n}{2}}$. Similarly, a vertex $(2, j_0)$ knows the pieces of information of vertices (i, j) , $i = 1, 2$ and $j = j_0, j_0 - 1, \dots, j_0 - 2^p + 1 \pmod{\frac{n}{2}}$.

As the last communication takes place through dimension 0, any vertex (i_0, j_0) will know the pieces of information of vertices (i, j) , $i = 1, 2$ and $j = j_0 - 2^p + 1 \pmod{\frac{n}{2}}, \dots, j_0 - 1, j_0, j_0 + 1, \dots, j_0 + 2^p - 1 \pmod{\frac{n}{2}}$. Which means that any vertex (i_0, j_0) knows the pieces of information of all the other vertices in the graph iff $j_0 + 2^p \pmod{\frac{n}{2}} \geq j_0 - 2^p + 1 \pmod{\frac{n}{2}}$, that is $n \leq 2^{p+2} - 2$. Hence this gossip scheme is valid for all n not a power of two and thus for $3 \times 2^p - 2 \leq n \leq 2^{p+2} - 2$, $W_{p+1,n}$ is a gossip graph. Consequently, we get $G(n) \leq \frac{n \times (p+1)}{2}$, where $p = g_n - 2$.

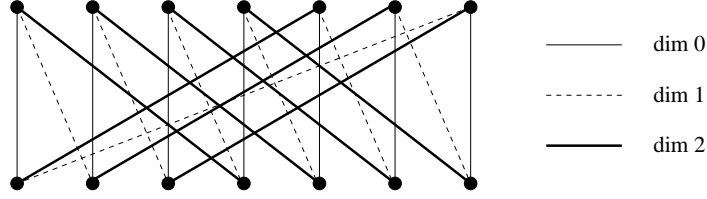


Figure 32: $W_{3,14}$ is a gossip graph

9.8 Proof of Theorem 12, page 9

Suppose we have $2^p + 1 \leq n \leq 2^{p+1} - 5$. Then $g_n = p + 2$. We will need to distinguish two cases depending on the value of n : $n = 4 \times m - 1$ and $n = 4 \times m + 1$.

9.8.1 $n = 4 \times m - 1$

Let us then take a Knödel graph on $2m$ vertices. We know that this graph can gossip in $p = g_n - 2$ time units. Let us then add $(2m - 1)$ pendent edges. If we valuate those edges with time units 1 and $(p + 2)$ while the Knödel graph gossips between time units 2 and $(p + 1)$, we get a gossip graph. The number of edges depends on the Knödel graph considered, which depends itself on the number n of vertices. As seen in Theorem 8, we have :

- If $2m \leq 3 \times 2^{p-2} - 4$, that is $n \leq 3 \times 2^{p-1} - 9$, then we can use the Knödel graph $W_{p-2,2m}$, which has $m \times (p - 2)$ edges. Thus in that case $G(n) \leq m \times (p - 2) + (2m - 1)$, that is $G(n) \leq m \times p - 1$.
- If $2m \geq 3 \times 2^{p-2} - 2$, that is $n \geq 3 \times 2^{p-1} - 5$, then we have to use the Knödel graph $W_{p-1,2m}$, which has $m \times (p - 1)$ edges. Thus $G(n) \leq m \times (p + 1) - 1$.

9.8.2 $n = 4 \times m + 1$

We use the same method as previously, except that we take a Knödel graph with $2(m + 1)$ vertices and that we add $(2m - 1)$ pendent edges. Similarly, the maximum degree Δ of the considered Knödel graph will depend on the value of n .

- If $2(m + 1) \leq 3 \times 2^{p-2} - 4$, that is $n \leq 3 \times 2^{p-1} - 11$, then we can use the Knödel graph $W_{p-2,2(m+1)}$, which has $(m + 1) \times (p - 2)$ edges. Thus $G(n) \leq (m + 1) \times p - 3$.
- If $2(m + 1) \geq 3 \times 2^{p-2} - 2$, that is $n \geq 3 \times 2^{p-1} - 7$, then we need to use the Knödel graph $W_{p-1,2(m+1)}$, which has $(m + 1) \times (p - 1)$ edges. Consequently, $G(n) \leq (m + 1) \times (p + 1) - 3$.

9.9 Proof of Theorem 16, page 10

The aim here is to prove the following lemma :

Lemma 1 *In a MGG_n where n is even, if there are s vertices of degree at least 3, then there are at most s vertices of degree 2.*

Proof : We recall that if n is even, there is no vertex of degree 1. Moreover, concerning vertices of degree 2, we know that :

- For $n \neq 2^{g_n-1} + 2$, if there is a vertex A of degree 2 adjacent to B and C , then B and C are of degree at least 3 and $\{1, g_n\} \subseteq AB \cup AC$ (cf. Theorems 2 and 5).
- If $n = 2^{g_n-1} + 2$, then if a vertex A is of degree 2 :

1. If one of its neighbours B is of degree 2, then let's denote by A_1 (resp. B_1) the vertex adjacent to A (resp. B) which is not B (resp. A). Then $AA_1 = BB_1 = \{1, g_n\}$ (cf. Proof of Theorem 5) ;
2. If none of the neighbours B and C of A is of degree 2, then $\{1, g_n\} \subseteq AB \cup AC$ (cf. Theorem 2).

Let us show now the lemma. We denote by V_2 the set of vertices of degree 2 in the graph and by V_{3+} the set of vertices of degree at least 3. Clearly $V = V_2 \cup V_{3+}$. Let Φ be the following function :

$$\begin{aligned} \Phi : V_2 &\longrightarrow V_{3+} \\ A &\longmapsto B \text{ such that } 1 \in AB. \end{aligned}$$

First, Φ is a function, that is each A in V_2 has an image in V_{3+} . Moreover, due to the 1-port model, Φ is injective, that is if $A \neq A'$, then $\Phi(A) \neq \Phi(A')$. Hence $|V_2| \leq |V_{3+}|$.

Let us now prove the theorem, that is $G(n) \geq \lceil \frac{5n}{4} \rceil$. As there is no vertex of degree 1 when n is even in a MGG_n , we will have vertices of degree 2 and vertices of degree 3 or more. Suppose we have s vertices of degree at least 3 in a MGG_n , that is we have $(n - s)$ vertices of degree 2. Lemma 1 yields that $n - s \leq s$. Moreover, we know that the sum of the degrees is twice the number of edges. Then, $2 \times G(n) \geq 3 \times s + 2 \times (n - s)$, that is $2 \times G(n) \geq 2 \times n + s$. Thanks to the first inequality, we get $G(n) \geq \frac{5n}{4}$. As $G(n)$ is an integer, we get the result.

9.10 Proof of Theorem 17, page 10

Let n be an even number such that $2^p - 2^d + 2 \leq n \leq 2^p$, with $p \geq d + 2 \geq 5$. First, as $n > 2^p - 2^d$, Theorem 3 yields that there cannot be any vertex of degree less or equal to $(p - d)$. Consequently, the minimum degree will be $(p - d + 1)$. Now let us prove the following lemma :

Lemma 2 *In a MGG_n such that $2^p - 2^d + 2 \leq n \leq 2^p$ with n even and $p \geq d + 2 \geq 5$, if there is a vertex A of degree $(p - d + 1)$, then it is incident to an edge AB such that $1 \in AB$ and B is of degree at least $(p - d + 2)$.*

Proof : Suppose we have a vertex A of degree $(p - d + 1)$. We recall that, for such values of n , $g_n = p$. First let us show that 1 (resp. 2) necessarily valuate one of the edges incident to A . Suppose that 1 does not valuate any of the edges incident to A . Then, the MBT rooted in A , as shown in Figure 33, holds $|V(MBT)| = 2^{p-1} - 2^{d-2} + 1$ vertices, which is strictly less than n for any $p \geq d + 1$ (we recall that we supposed $p \geq d + 2$). Hence 1 valuates one of the edges incident to A . Let us denote this edge AB .

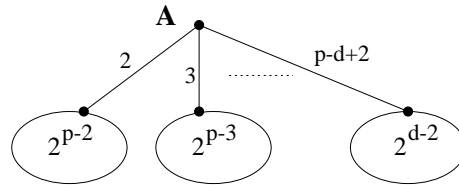


Figure 33: MBT rooted in A

Similarly, suppose 2 does not valuate any of the edges incident to A . Then the MBT rooted in A , as shown in Figure 34, holds $|V(MBT)| = 3 \times 2^{p-2} - 2^{d-2} + 1$ vertices, which is strictly less than n for any $p \geq d + 2$. Hence 2 valuates one of the edges incident to A . Symmetrically, we can show thanks to the RBT s rooted in A that p and $(p - 1)$ necessarily valuate the edges incident to A .

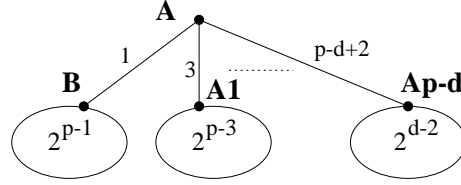


Figure 34: *MBT* rooted in A

Now let us consider the vertex B such that $1 \in AB$, and let us prove that B is of degree at least $(p - d + 2)$. Suppose B is of degree $(p - d + 1)$. Then the *MBT* rooted in A , as shown in Figure 35, holds exactly $n = 2^p - 2^d + 2$ vertices.

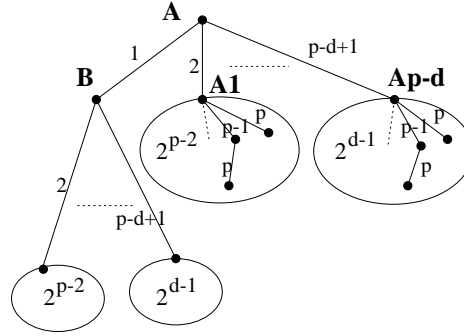


Figure 35: *MBT* rooted in A

However, in that case all of the vertices $A_1 \dots A_{p-d}$ are such that $(p - 1)$ and p value their incident edges, otherwise the *MBT* rooted in A would not hold enough vertices. Moreover, $d \geq 3$, which means that p and $(p - 1)$ are distinct from any of the numbers $1, 2 \dots, p - d + 1$. Consequently, due to the 1-port model, the only edge incident to A that can be valued by p as well as $(p - 1)$ is the edge AB . But both p and $(p - 1)$ valuating AB is pointless and leads to a *RBT* rooted in A with strictly less than n vertices.

We have then shown that if there is a vertex A of degree $(p - d + 1)$, then it is adjacent to the vertex B of degree at least $(p - d + 2)$ such that $1 \in AB$. Now let us prove the following lemma :

Lemma 3 *In a MGG_n such that $2^p - 2^d + 2 \leq n \leq 2^p$ with n even and $p \geq d + 2 \geq 5$, if there are s vertices of degree $(p - d + 1)$, then there are at least s vertices of degree at least $(p - d + 2)$.*

Proof : Note that this proof is very close to the proof of Lemma 1. Let V_1 be the set of vertices of degree $(p - d + 1)$ in the graph and V_2 the set of vertices of degree at least $(p - d + 2)$. Clearly $V = V_1 \cup V_2$. Let Φ be the following function :

$$\Phi : V_1 \longrightarrow V_2$$

$$A \longmapsto B \text{ such that } 1 \in AB.$$

First, Φ is a function, that is each A in V_1 has an image in V_2 (Lemma 2). Moreover, due to the 1-port model, Φ is injective, that is if $A \neq A'$, then $\Phi(A) \neq \Phi(A')$. Hence $|V_1| \leq |V_2|$.

Finally, let us prove the theorem. We know that in a MGG_n such that $2^p - 2^d + 2 \leq n \leq 2^p$ with n even and $p \geq d + 2 \geq 5$:

- The minimum degree is $(p - d + 1)$;

- If there are s vertices of degree $(p - d + 1)$, then there are at least s vertices of degree at least $(p - d + 2)$.

Hence let us suppose we have s vertices of degree $(p - d + 1)$ in a MGG_n . Then the $(n - s)$ remaining vertices are of degree at least $(p - d + 2)$ and Lemma 3 yields $n - s \geq s$ (1). Moreover, the sum of the degrees is twice the number of edges. That is : $2 \times G(n) \geq s \times (p - d + 1) + (n - s) \times (p - d + 2)$ (2). Combining (1) and (2) leads to the asserted formula, that is : $G(n) \geq \lceil \frac{n \times (2p - 2d + 3)}{4} \rceil$.

9.11 Proof of Theorem 19, page 10

Suppose $G(n) = n$. Then, we know that there can only be vertices of degree 2 and/or vertices contributing to (1,3)-type edges. We distinguish two cases :

- If there is no (1,3)-type edge, then the considered graph is C_n , cycle of length n . Consequently, $D(C_n) = \lfloor n/2 \rfloor$. As $\lfloor n/2 \rfloor > \lceil \log_2(n) \rceil + 1$ for any $n \geq 13$, C_n cannot be MGG_n for any $n \geq 13$.
- If there is at least one (1,3)-type edge AB , then we show, thanks to Figure 36, that the MBT rooted in A holds at most $|V(MBT)| = 4 \times g_n - 8$ vertices, whatever the number of (1,3)-type edges is. Yet, for any $n \geq 13$, $n > |V(MBT)|$, which shows that case is impossible too.

Consequently, $G(n) \neq n$ and $G(n) \geq n + 1$.

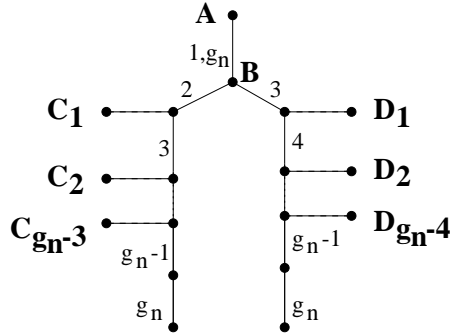


Figure 36: MBT rooted in A where AB is a (1,3)-type edge

9.12 Proof of Theorem 20, page 10

The construction of the proof is the following : first, we prove Properties 6 and 7 below. These properties are necessary to prove Lemma 4. Finally, Lemma 4 will allow us to prove the theorem.

Property 6 For any n odd such that $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, if there are two adjacent vertices of degree 2, A and B , such that A is also adjacent to A_1 and B is also adjacent to B_1 , then :

- If $1 \in AB$, then $2 \in AA_1$ and $2 \in BB_1$;
- If $g_n \in AB$, then $(g_n - 1) \in AA_1$ and $(g_n - 1) \in BB_1$.
- Moreover, in the case $n = 2^p - 1$, we cannot have $1 \in AB$ and $g_n \in AB$.

Remark : We recall that in that case $g_n = p + 1$.

Proof : Let's consider two adjacent vertices of degree 2, A and B , as described above. First we recall that Theorem 2 yields that 1 and g_n value the edges adjacent to A and B , so that if $1 \notin AB$, necessarily $1 \in AA_1$ and $1 \in BB_1$. Notice that the same goes to g_n .

Suppose now we have $1 \in AB$. If $2 \notin AA_1$ and $2 \notin BB_1$, we get a MBT rooted in A with $|V(MBT)| = 2^{g_n-2} + 2$ vertices, as shown in the left figure of Figure 37. However, $|V(MBT)| < n$ for any $g_n \geq 4$.

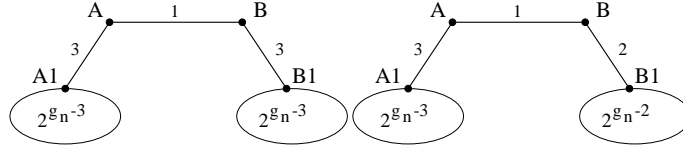


Figure 37: MBT s rooted in A

Suppose now, w.l.o.g., that $2 \notin AA_1$ and $2 \in BB_1$. The MBT rooted in A holds $|V(MBT)| = 3 \times 2^{g_n-3} + 2$ vertices, as shown in the right figure of Figure 37. Now consider the valuation g_n . If $g_n \in AB$, then, by a symmetrical proof on RBT s, we show that necessarily either the couple $((g_n - 1), (g_n - 1))$, $((g_n - 1), (g_n - 2))$ or $((g_n - 2), (g_n - 1))$ values the edges (AA_1, BB_1) . In any case, the MBT rooted in A taking into account these constraints holds at most $|V(MBT)| = 3 \times 2^{g_n-3}$ vertices, which is always strictly less than n (Figures (a), (b) and (c) in Figure 38).

Moreover, if $g_n \notin AB$, then we know that $g_n \in AA_1$ and $g_n \in BB_1$, and the MBT rooted in A holds $|V(MBT)| = 3 \times 2^{g_n-3}$ vertices, which is strictly less than n too (cf. Figure (d) of Figure 38). So that if $1 \notin AB$, then $2 \in AA_1$ and $2 \in BB_1$.

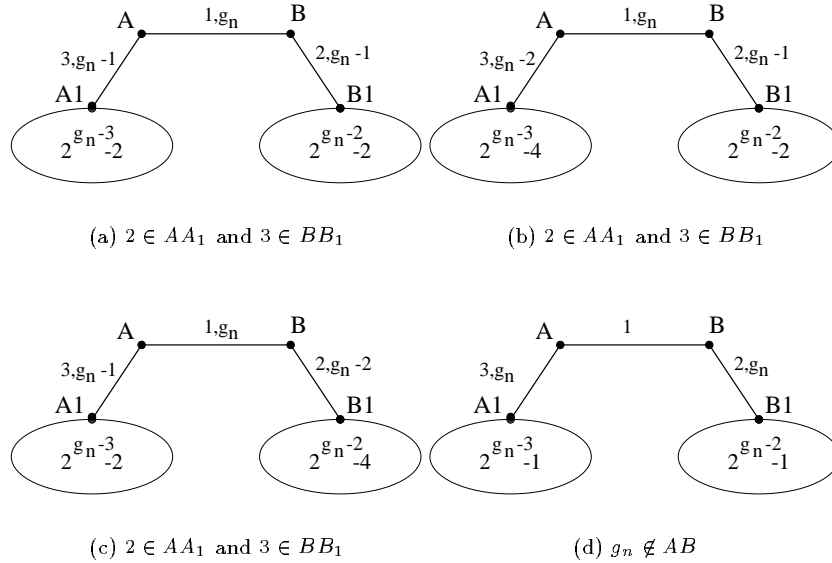


Figure 38: MBT s rooted in A

Now suppose $g_n \in AB$, and let's consider the valuation 1. If $1 \in AB$, we have proved that necessarily $2 \in AA_1$ and $2 \in BB_1$. W.l.o.g., suppose that $(g_n - 1) \notin AA_1$. Then the RBT rooted in A holds $|V(RBT)| = 3 \times 2^{g_n-3} - 2$ vertices for any $g_n \geq 5$, and $|V(RBT)| < n$ in any case (cf. Figure (a) of Figure 39). If $1 \notin AB$, then $1 \in AA_1$ and $1 \in BB_1$ and we get a RBT rooted in A with $|V(RBT)| = 3 \times 2^{g_n-3}$ vertices, which is also strictly less than n (cf. Figure (b) of Figure 39). The proof is then complete for $3 \times 2^{g_n-3} + 1 \leq n \leq 2^{g_n-1} - 1$.

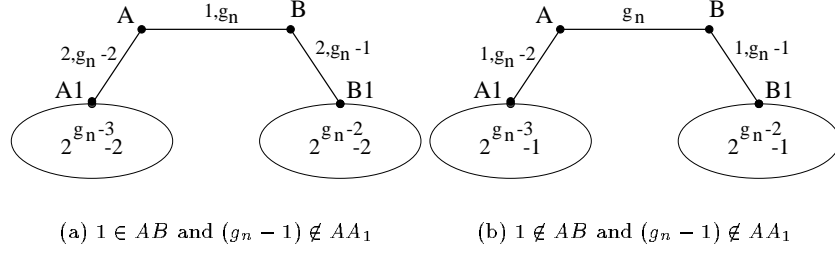


Figure 39: RBT s rooted in A

Finally, in the case $n = 2^p - 1$, let's suppose we have $\{1, g_n\} \subseteq AB$. Then the first part of the property yields $\{2, g_n - 1\} \subseteq AA_1$ and $\{2, g_n - 1\} \subseteq BB_1$. Then the MBT rooted in A , as shown in Figure 40 holds $|V(MBT)| = 2^{g_n-1} - 2$ vertices, while $n = 2^{g_n-1} - 1$. Hence $|V(MBT)| < n$, which shows 1 and g_n cannot value AB at the same time.

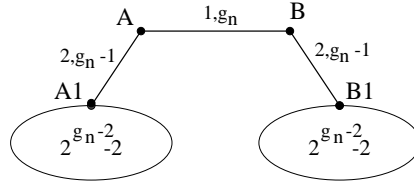


Figure 40: MBT rooted in A where $\{1, g_n\} \subseteq AB$

Property 7 For n odd such that $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, if there is a $(1, q)$ -type edge AB in a MGG_n such that B is of degree q adjacent to C of degree 2 and C is adjacent to D , then :

- $\{1, g_n\} \subseteq CD$;
- D cannot be of degree 2.

Proof : The first part of the lemma comes from Theorem 1, which yields that if an edge AB is of type $(1, q)$, then $\{1, g_n\} \subseteq AB$. Consequently, as we consider a 1-port model, 1 and g_n cannot value BC . Yet, C is of degree 2 and Theorem 2 says that $\{1, g_n\} \subseteq BC \cup CD$, hence the result.

Now let's show that D cannot be of degree 2. If we suppose $\deg(D) = 2$, as $\{1, g_n\} \subseteq CD$, Property 6 yields that $\{2, g_n - 1\} \subseteq BC$ and $\{2, g_n - 1\} \subseteq DE$, where E is the vertex adjacent to D which is not C , as shown in Figure 41. Hence, one can notice that $q \leq g_n - 2$. Note that in the case $g_n = 5$, this could not happen since $q \geq 4$ (cf. Theorem 4) and $q \leq g_n - 2$, as stated above.

If we now look at the MBT rooted in A , we show that it holds $|V(MBT)| = 5 \times 2^{g_n-4} - 2^{g_n-q} + 2$ vertices, as shown in Figure 41. As $q \leq g_n - 2$, $|V(MBT)| \leq 5 \times 2^{g_n-4} - 2$, which is strictly less than n . Hence, D cannot be of degree 2.

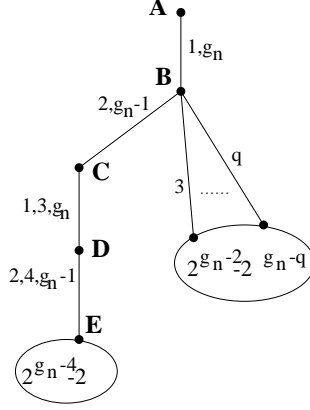


Figure 41: MBT rooted in A

Lemma 4 For all n odd such that $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, if there are s vertices of degree at least 3 in a MGG_n (vertices incident to $(1, q)$ -type edges excluded), then there are at most $2s$ vertices of degree 2.

Proof : In order to prove this lemma, we need to recall certain properties concerning MGG_n for such values of n . Those properties are the following :

- If there is a vertex of degree 2 in a MGG_n then 1 and g_n valuate its incident edges (cf. Theorem 2) ;
- A vertex of degree 2 is adjacent to at most 1 vertex of degree 2 (cf. Theorem 6) ;
- If two vertices of degree 2, A and B , are adjacent, let's call A_1 (resp. B_1) the vertex adjacent to A (resp. B) which is not B (resp. A). Then we get, thanks to Property 6 :
 1. If $1 \in AB$, then $2 \in AA_1$ and $2 \in BB_1$ (we recall that if $1 \notin AB$, then Theorem 2 yields $1 \in AA_1$ and $1 \in BB_1$) (constraint **C1**) ;
 2. If $g_n \in AB$, then $(g_n - 1) \in AA_1$ and $(g_n - 1) \in BB_1$ (we recall that if $g_n \notin AB$, then Theorem 2 yields $g_n \in AA_1$ and $g_n \in BB_1$).
- There cannot be any $(1,3)$ -type edge (cf. Theorem 4) ;
- If there is a $(1, q)$ -type edge AB , with $q \geq 4$, such that A is of degree 1, then if B is adjacent to C of degree 2, and C is adjacent to D , then we know, thanks to Property 7 :
 1. $\{1, g_n\} \subseteq CD$;
 2. D cannot be of degree 2 (constraint **C2**).

We denote by V_2 the set of vertices of degree 2, and V_{3+} the set of vertices of degree at least 3. V_2 can be decomposed in three parts :

- $V_{2,a}$ is the set of vertices of degree 2 adjacent to vertices of degree at least 3;
- $V_{2,b}$ is the set of vertices of degree 2 such that any vertex A of this set has a neighbour B of degree 2 and $1 \notin AB$;

- $V_{2,c}$ is the set of vertices of degree 2 such that any vertex A of this set has a neighbour B of degree 2 and $1 \in AB$;

Clearly, $V_2 = V_{2,a} \oplus V_{2,b} \oplus V_{2,c}$. Analogously we can decompose V_{3+} into two distinct subsets, which are $V_{1,q}$ and $\overline{V_{1,q}}$, where $V_{1,q}$ is the set of vertices of degree at least 3 having a neighbour of degree 1, and $\overline{V_{1,q}}$ is its complement in V_{3+} .

The properties stated above ensure us that Φ and Ψ , defined below, are functions, that is any vertex A has an image. Φ and Ψ are as follows :

$$\begin{aligned} \Phi : V_{2,a} \cup V_{2,b} &\longrightarrow \overline{V_{1,q}} \\ A &\longmapsto B \text{ such that } 1 \in AB. \end{aligned}$$

Clearly, B is in $\overline{V_{1,q}}$ because a vertex in $V_{1,q}$ already has the valuation 1 on the $(1, q)$ -type edge. Moreover, Φ is injective because of the 1-port hypothesis.

$$\begin{aligned} \Psi : V_{2,c} &\longrightarrow \overline{V_{1,q}} \\ A &\longmapsto B \text{ such that } 1 \notin AB. \end{aligned}$$

Note that B cannot be in $V_{1,q}$, following constraint **C2** ; which implies $B \in \overline{V_{1,q}}$. Moreover, Ψ is injective because if $1 \notin AB$, then constraint **C1** yields that $2 \in AB$, and because we consider a 1-port model. As Ψ and Φ are injective, we get the following inequalities :

1. $|V_{2,a}| + |V_{2,b}| \leq |\overline{V_{1,q}}|$;
2. $|V_{2,c}| \leq |\overline{V_{1,q}}|$.

As the subsets of V_2 are distinct and their union is V_2 itself, adding those inequalities we get $|V_2| \leq 2 \times |\overline{V_{1,q}}|$. Hence the result.

Now, let's prove the main theorem, that is $G(n) \geq \lceil \frac{7n}{6} \rceil$. Let's suppose we have r $(1, q)$ -type edges, where q may be variable. Let's suppose we have s vertices of degree at least 3, such that those vertices haven't got an incident $(1, q)$ -type edge. Then there are $(n - s - 2r)$ vertices of degree 2. Analogously to the proof of Theorem 16, we get two inequalities :

1. $n - s - 2r \leq 2s$;
2. $2 \times G(n) \geq 3s + 5r + 2 \times (n - s - 2r)$.

Those inequalities lead to the following one : $G(n) \geq \frac{7n+r}{6}$. At worst, $r = 0$, hence the result.

9.13 Proof of Theorem 21, page 11

In this section, we are going to prove Property 8 and Lemmas 5, 6 and 7, which will help us to prove the theorem.

Property 8 *For $n = 2^p - 1$ with $p \geq 3$, if A is a vertex of degree at least 3, adjacent to B and C both of degree 2 such that B (resp. C) is adjacent to A and B_1 (resp. A and C_1), then we cannot have $\deg(B_1) = \deg(C_1) = 2$.*

Proof : Let's suppose $\deg(B_1) = \deg(C_1) = 2$, as shown in Figure 42. As $n = 2^p - 1$, following Property 6 we know that 1 and g_n do not valueate BB_1 (resp. CC_1) at the same time. But if neither 1 nor g_n valueate, say, BB_1 , then they both must valueate AB (cf. Theorem 2). As we considered a 1-port model, in that case 1 and g_n don't valueate AC . As a consequence, following Theorem 2, 1 and g_n must valueate CC_1 , hence there is a contradiction. Then the only possible cases are when $1 \in BB_1$ and $g_n \in CC_1$, or the other way round. Suppose, w.l.o.g., that $1 \in BB_1$ and $g_n \in CC_1$. Hence, Property 6 implies $\{2, g_n\} \subseteq AB$, $\{2, g_n\} \subseteq B_1B_2$, $\{1, g_n - 1\} \subseteq AC$ and $\{1, g_n - 1\} \subseteq C_1C_2$ (we refer to Figure 42 for a better understanding of these statements). Then the *MBT* rooted in A holds $|V(\text{MBT})| = 7 \times 2^{g_n-4} - 2$ vertices, which is strictly less than $n = 2^{g_n} - 1$ for any $g_n \geq 4$. As a consequence we cannot have both B_1 and C_1 of degree 2.

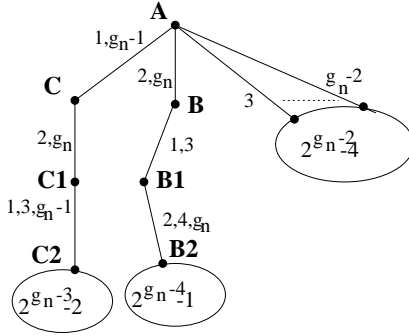


Figure 42: MBT rooted in A

Lemma 5 For all $n = 2^p - 1$ with $p \geq 4$, the only $(1, q)$ -type edge that can exist in a MGG_n is for $q = p$.

Proof : Theorem 1 yields that if AB is a $(1, q)$ -type edge, then $AB = \{1, g_n\}$. Therefore, as $g_n = p + 1$ and AB has 2 valuations, we have $q \leq p$ (we recall that the edge AB is taken into account in the value of q).

Suppose we have a $(1, q)$ -type edge AB , and let's call the vertices adjacent to B (A excluded) $B_2 \dots B_q$. Now let's build the MBT rooted in A , as shown in Figure 43. Clearly, it holds $|V(MBT)| = 2^{g_n-1} - 2^{g_n-q} + 2$ vertices. As $q \neq g_n$ since $g_n \in AB$, the only case for which $|V(MBT)| \geq n$ is when $q = g_n - 1$ (and, in that case, $|V(MBT)| = 2^{g_n-1} = 2^p$). Since $g_n = p + 1$, we get the result.

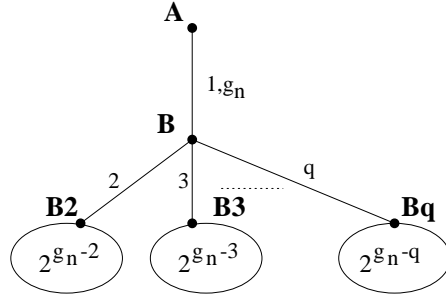


Figure 43: MBT rooted in A where AB is a $(1, q)$ -type edge

Lemma 6 For all $n = 2^p - 1$ with $p \geq 4$, if we have a $(1, p)$ -type edge AB in a MGG_n such that A is of degree 1, then B can't be adjacent to any vertex of degree 2.

Proof : Let's consider a $(1, p)$ -type edge, and suppose that B_j is of degree 2. Let's call C_j the vertex adjacent to B_j which is not B , as shown in Figure 44. Then Theorem 2 yields that 1 and g_n valuate $B_j C_j$. If we then consider the MBT rooted in A , we get $|V(MBT)| = 2^{g_n-1} - 2^{g_n-j-1} + 1$, which is strictly less than n for any $j \neq g_n - 1$.

In the case $j = g_n - 1$, however, considering the RBT rooted in A as shown in Figure 45, we get $|V(RBT)| = 3 \times 2^{g_n-3}$, which is strictly less than n for any $g_n \geq 4$.

In all cases, we show that there is at least a RBT or a MBT that cannot hold n or more vertices. Then there is no vertex of degree 2 adjacent to B in a MGG_n such that $n = 2^p - 1$.

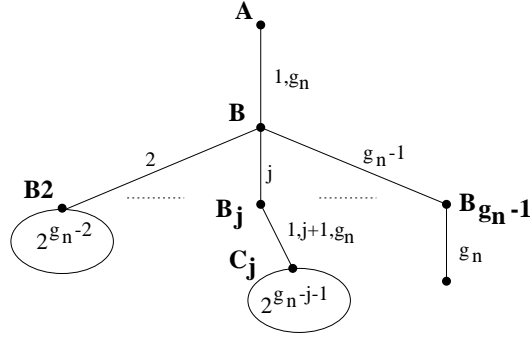


Figure 44: *MBT* rooted in A , where B_j is of degree 2

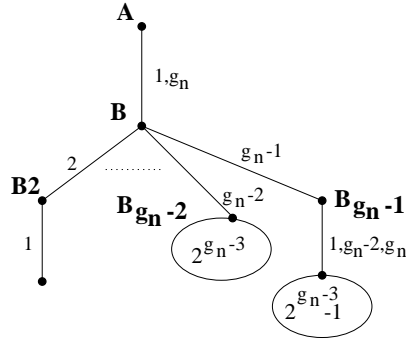


Figure 45: *RBT* rooted in A where B_{g_n-1} is of degree 2

Lemma 7 For all $n = 2^p - 1$ with $p \geq 4$, if there are s vertices of degree at least 3 in a MGG_n (vertices incident to $(1, p)$ -type edges excluded), then there are at most $\frac{3s}{2}$ vertices of degree 2.

Proof : We need first to recall different properties concerning MGG_n for such a value of n . They are the following :

- If there are $(1, q)$ -type edges, then necessarily $q = p$ (cf. Lemma 5) ;
- If there is a $(1, p)$ -type edge AB where B is of degree p , then B cannot be adjacent to any vertex of degree 2 (cf. Lemma 6) ;
- If there is a vertex A of degree 2, adjacent to B and C , then $\{1, g_n\} \subseteq AB \cup AC$ (cf. Theorem 2) ;
- If A is of degree 2, adjacent to B and C , then we cannot have $\deg(B) = \deg(C) = 2$ (cf. Theorem 6) ;
- If two vertices of degree 2, A and B , are adjacent, let's call A_1 (resp. B_1) the vertex adjacent to A (resp. B) which is not B (resp. A). Then we get, thanks to Property 6 :
 1. If $1 \in AB$, then $2 \in AA_1$ and $2 \in BB_1$ (we recall that if $1 \notin AB$, then Theorem 2 implies $1 \in AA_1$ and $1 \in BB_1$) ;
 2. If $g_n \in AB$, then $(g_n - 1) \in AA_1$ and $(g_n - 1) \in BB_1$ (we recall that if $g_n \notin AB$, then Theorem 2 implies $g_n \in AA_1$ and $g_n \in BB_1$) ;

3. We cannot have $1 \in AB$ and $g_n \in AB$.

- If A is a vertex of degree at least 3 adjacent to two vertices of degree 2, B and C , such that B (resp. C) is adjacent to A and B_1 (resp. A and C_1), then we cannot have $\deg(B_1) = \deg(C_1) = 2$ (constraint **C1**, cf. Property 8).

Let V_2 denote the set of vertices of degree 2 and V_{3+} the set of vertices of degree at least 3. Let us decompose V_2 and V_{3+} as it has been done in Lemma 4, and let us define the following functions :

$\Phi : V_{2,c} \longrightarrow W_1$

$A \longmapsto B$ such that A is adjacent to a vertex $A' \neq B$ of degree 2 and $g_n \in AB$.

$\Psi : V_{2,b} \longrightarrow W_2$

$A \longmapsto B$ such that A is adjacent to a vertex $A' \neq B$ of degree 2 and $1 \in AB$.

Clearly, W_1 and W_2 are subsets of V_{3+} . Moreover, $W_1 \cap W_2 = \emptyset$, thanks to constraint **C1**.

Let us then define two more functions :

$\Sigma : V_{2,a} \longrightarrow \overline{V_{1,q}} - W_2$

$A \longmapsto B$ such that $1 \in AB$.

$\Sigma' : V_{2,a} \longrightarrow \overline{V_{1,q}} - W_1$

$A \longmapsto B$ such that $g_n \in AB$.

Clearly, any $B = \Sigma(A)$ is in $\overline{V_{1,q}} - W_2$, because otherwise two edges incident to a vertex of degree at least 3 would be valued by 1. Analogously, any $B = \Sigma'(A)$ is in $\overline{V_{1,q}} - W_1$.

First, note that any vertex A has an image by any of the functions above, provided it is in the right subset. Moreover, Φ , Ψ , Σ and Σ' are all injective, thanks to the 1-port model. We then get two inequalities, depending whether we look at Σ or Σ' :

1. $|V_2| \leq |\overline{V_{1,q}}| + |W_1|$;

2. $|V_2| \leq |\overline{V_{1,q}}| + |W_2|$.

If we add them, we get $2 \times |V_2| \leq 2 \times |\overline{V_{1,q}}| + |W_1| + |W_2|$. As $|W_1| + |W_2| \leq |\overline{V_{1,q}}|$, we get $2 \times |V_2| \leq 3 \times |\overline{V_{1,q}}|$, hence the result.

Finally, let us come back to the main result, that is $G(n) \leq \lceil \frac{6n}{5} \rceil$. Let's suppose that we have r $(1, p)$ -type edges, and s vertices of degree at least 3 (vertices incident to $(1, p)$ -type edges excluded). Consequently, we have $(n - s - 2r)$ vertices of degree 2. Hence the first inequality : $n - s - 2r \leq \frac{3s}{2}$. Moreover, the sum of the vertices degrees is twice the number of edges, that is $2 \times G(n) \geq 3s + (p + 1) \times r + 2 \times (n - s - 2r)$. Finally, we get the following inequality : $G(n) \geq \frac{6n}{5} + \frac{5(p+1)-24}{10} \times r$. We recall that $g_n = p + 1$ in that case. At worst, there is no $(1, p)$ -type edge and we get the result.

9.14 Proof of Theorem 23, page 11

We already know that $G(15) \geq 18$ thanks to Theorem 21, and that $G(15) \leq 19$ thanks to Corollary 3. Let us recall to those two properties, coming from Lemma 7 :

1. If there are q vertices of degree at least 3, vertices incident to $(1, 4)$ -type edges excluded, then there can be at most $\frac{3q}{2}$ vertices of degree 2 ;
2. $G(n) \geq 18 + \frac{r}{10}$, where r is the number of $(1, 4)$ -type edges.

In that case, suppose $G(15) = 18$. Then there is no $(1, 4)$ -type edge. As the only vertices of degree 1 that can exist are vertices incident to $(1, 4)$ -type edges, it means that we cannot have any vertex of degree 1. Hence, if we have s vertices of degree at least 3, then we have exactly $(n - s)$ vertices

of degree 2. Moreover, we know that $n - s \leq \frac{3s}{2}$ (Lemma 7), that is $s \geq 6$. As we supposed $G(15) = 18$, the only possible case is when we have 6 vertices of degree 3 and 9 vertices of degree 2.

Let's now consider that case, and recall the proof of Lemma 7, where we had those two inequalities :

- $|V_2| \leq |\overline{V_{1,q}}| + |W_1|$;
- $|V_2| \leq |\overline{V_{1,q}}| + |W_2|$.

Those inequalities imply $|W_i| \geq 3$ with $i \in \{1, 2\}$. As those two subsets are distinct and $|\overline{V_{1,q}}| = 6$, we get $|W_1| = |W_2| = 3$. However $|V_{2,b}|$ and $|V_{2,c}|$ are even, by definition. We knew, thanks to the function Ψ , that $|V_{2,b}| \leq |W_2|$. We can then say that $|V_{2,b}| \leq |W_2| - 1$. Moreover, $|V_2| = |V_{2,a}| + |V_{2,b}| + |V_{2,c}|$. Hence we have, thanks to Ψ , Φ and Σ , the following inequality : $|V_2| \leq |\overline{V_{1,q}}| + |W_1| - 1$, that is $9 \leq 8$. Then this case cannot lead to a MGG_{15} ; therefore $G(15) \neq 18$ and $G(15) = 19$.