A Study of Minimum Gossip Graphs

Guillaume FERTIN

LaBRI U.R.A. CNRS 1304, Université Bordeaux I
351 Cours de la Libération, F33405 Talence Cedex

fertin@labri.u-bordeaux.fr

Abstract

Gossiping and broadcasting are two problems of information dissemination described in a group of individuals connected by a communication network. In broadcasting, one individual has an item of information and needs to communicate it to everyone else. In gossiping, every person in the network knows a unique item of information and needs to communicate it to everyone else. Those two communication patterns find their main applications in the field of interconnection networks for parallel and distributed architecture.

After reviewing some of the main results that have been obtained on these problems, we give new properties concerning gossiping, mainly about the structure of the networks gossiping in minimum time and with a minimum number of communication links $G(n)$. These properties lead to new bounds for $G(n)$ in the general case, and in particular when $1 \leq n \leq 32$. Moreover, for some values of n (namely $n = 15$, $n = 24$ and $n = 28$), we show the exact value of $G(n)$.

1 Introduction

This paper presen s new resul s on graphs performing gossiping in minimum ime and wi h a minimum number of edges. Those new resul s mainly concern he s ruc ure of such graphs, and consequen ly give new bounds for heir number of edges.

In he nex sec ion, we give some defini ions in roducing broadcas ing and gossiping. In he hird sec ion, we give some o her defini ions and no a ions necessary o ackle he ques ion. Sec ion 4 provides a quick survey of broadcas ing and gossiping. Sec ion 5 is devo ed o he s ruc ure of he concerned graphs, and he wo following sec ions will provide respec ively a series of upper and lower bounds concerning he number of edges of such graphs. Finally, sec ion 8 presen s new resul s and/or bounds for graphs having be ween 1 and 32 ver ices, as well as a quick sum-up of he proper ies of minimum gossip graphs.

2 Definitions

We will use, as far as possible in he following, he same defini ions and no a ions han hose given in [dR94, HHL88, Lei92]. A parallel - or dis ribu ed - archi ec ure will be modelled by a graph, where ver ices will represen processors - or compu ers -, and edges will represen communica ion links. We shall consider a *constant time*, *1-port* and *full-duplex* model, e.g. :

- each message sen from one node o i s neighbour akes one ime uni ;
- each ver ex can only communica e wi h one of i s neighbours a he same ime ;
- if an edge joins a ver ex u o a ver ex v, hen he communica ion akes place bo h from u o v and from v o u.

Such a model is represen ed by undirec ed and connec ed graphs wi hou loops or mul iple edges.

2.1 Broadcasting

Broadcas ing, also called *one-to-all*, refers o sending an i em of informa ion from one par icular node o every o her in he graph. The erm $b(G)$ will deno e he minimum amoun of ime necessary o broadcas in graph G from any ver ex v, or he *broadcast time* of G. If one considers he comple e graph K_n, i is qui e obvious ha $b(K_n) = \lceil \log_2 n \rceil$, since he number of informed ver ices can a mos double every ime uni . Ilog log be his value of B M B a G) a b a)) a

3.2 Notions of MBT and RBT

In his sec ion, le us consider a graph G wi h n ver ices. Wha we wan o know is whe her his graph allows gossiping in t ime uni s or no . Mos of he ime, we know differen charac eris ics of G, such as he degree(s) of one or several ver ex (ver ices) of G, or he valua ion of some of i s edges, or even bo h.

3.2.1 t-Maximum Broadcast Trees (MBT)

In order for G o gossip in t ime uni s, i is necessary ha each ver ex of G is able o broadcas in t ime uni s, especially ver ices for which hypo hesis are made.

We hen call *t-Maximum Broadcast Tree*, or MBT, roo ed in A, a ree roo ed in A which allows o broadcas i s informa ion o a maximum number of ver ices in t ime uni s.

Example : Le 's suppose ha in a graph G of order 15, ha we'd like o be MGG_{15}, we have a ver ex A of degree 1 such ha he edge AB joins A o he res of he graph. Le 's suppose ha his edge is such ha $AB = \{1, 5\}$, and ha B is of degree 3. If we build he MBT roo ed in A, we ge he ree shown in Figure 1, which has 14 ver ices. Consequen ly, under hose hypo hesis, A can' broadcas in 5 ime uni s o he res of he graph. Which means ha a leas one of he above hypo hesis is wrong.

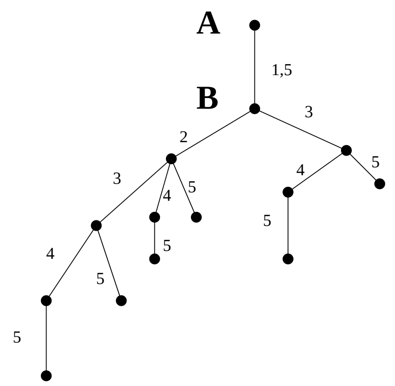

Figure 1: An example of MBT

3.2.2 Reverse Broadcast Trees (RBT)

In order for G o gossip wi hin t ime uni s, i is necessary ha all he ver ices of G can be informed of he con en of each o her ver ex in t ime uni s. This is also rue for he ver ices abou which some hypo hesis are made.

We hen call *Reverse Broadcast Tree*, or RBT roo ed in A, a ree roo ed in A, which, following he hypo hesis made abou he ver ices and edges of G, ga hers in A he informa ion of a maximum number of ver ices in t ime uni s. Moreover, if A_1A_2 is valua ed by $m_{A_1A_2}$ in he MBT, and by $r_{A_1A_2}$ in he corresponding RBT, we have he following equali y : $r_{A_1A_2} + m_{A_1A_2} = t + 1$.

Remark : a requisi e condi ion for a graph o be a MGG_n is ha i s MBT and RBT roo ed in each of i s ver ices all have n or more ver ices. However, his is obviously no a sufficien condi ion, because i doesn' ake in accoun nei her he number of edges of he graph nor he possible incompa ibili ies be ween he valua ions in each MBT and RBT.

4 Known results

4.1 Immediate result

Property 1 *For all n even, $G(n) \geq B(n)$.*

Proof : Wha ever he order of he graph is, gossiping requires broadcas ing. Therefore, as gossip and broadcas imes are he same for all n even, he number of edges in a MGG mus be superior o he number of edges in a MBG.

4.2 Broadcasting

Property 2 *[FHMP79] For all $n = 2^p$, $B(n) = p \times 2^{p-1}$.*

Property 3 *[DFF91] For all $n = 2^p - 2$ $(p \geq 3)$, $B(n) = (p-1) \times (2^{p-1} - 1)$.*

4.3 Gossiping

Property 4 *[Lab86] In a n-vertices tree T_n, the gossip time $g(T_n)$ is such that*

$$g(T_n) \geq 2 \times \lceil \log_2 n \rceil - 1.$$

Property 5 *[Lab93]*
For all $n = 2^p$, $G(n) = p \times 2^{p-1}$.
For all $n = 2^p - 2$ $(p \geq 4)$, $G(n) = (p-1) \times (2^{p-1} - 1)$.
For all $n = 2^p - 4$ $(p \geq 6)$, $G(n) = (p-1) \times (2^{p-1} - 2)$.

5 Structure of a MGG_n

5.1 Cut-edges in a MGG_n

Theorem 1 *In a MGG_n :*

1. *For all n even $(n \geq 4)$, there is no cut-edge.*
2. *For all n odd $(n \geq 5)$, the only cut-edges that can exist are pendent edges AB. Moreover, $AB = \{1, g_n\}$.*

Proof : Le AB be a cu -edge dividing he graph G in G_1 and G_2. Suppose ha $|AB| = \{t_1, \ldots, t_p\}$ where $1 \leq t_1 < t_2 \ldots < t_p \leq g_n$. Le t_i (resp. t_j) be he smalles valua ion on AB such ha A (resp. B) knows all he informa ion of G_1 (resp. G_2) wi hin $t_i - 1$ (resp. $t_j - 1$) ime uni s a mos . Le $a = |V(G_1)|$ and $b = |V(G_2)|$, where $a + b = n$, and le us suppose, w.l.o.g., ha $a \geq \lceil \frac{n}{2} \rceil$. Thanks o he broadcas imes of each G_i, we have he four following inequali ies :

- $t_i - 1 \geq \lceil \log_2 a \rceil$ and $g_n - t_i \geq \lceil \log_2 b \rceil$;
- $t_j - 1 \geq \lceil \log_2 b \rceil$ and $g_n - t_j \geq \lceil \log_2 a \rceil$.

Those inequali ies lead o he following one : $g_n \geq \lceil \log_2 a \rceil + \lceil \log_2 b \rceil + 1$. **(I1)**
Now le n be even, and le us consider wo cases :

- $b = 1$. In ha case, $a = n - 1$ and **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n-1) \rceil + 1$, which is false for any even $n \geq 4$.
- $2 \leq b \leq \frac{n}{2}$. In ha case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 n \rceil + 1$, which is false for any n even.

Hence, if n is even $(n \geq 4)$, here is no cu -edge in a MGG.
Le n be odd, and a and b as above. Le us dis inguish hree cases ;

- $b = 1$, ha is BA is a penden edge. Then **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n-1) \rceil$, which is rue for any n. In ha case, he four inequali ies above lead o wo cases :

  1. If $n \neq 2^p + 1$, we ge $t_j \leq 1$ and $t_i \geq g_n$;
  2. If $n = 2^p + 1$, we ge $t_j \leq 2$ and $t_i \geq g_n - 1$, which leads o four possible valua ion schemes for he penden edge BA. However, he MBTs and RBTs roo ed in B hold enough ver ices only in he case $BA = \{1, g_n\}$.

- $b = 2$. In ha case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n-2) \rceil + 1$, which is rue only for $n = 2^p + 1$. Le $V(G_1) = \{B, B_1\}$. Then BB_1 is a penden edge and, as seen above, we have $BB_1 = \{1, g_n\}$. Moreover, he firs four inequali ies yield $t_j = 2$ and $t_i = g_n - 1$. However, he MBT roo ed in B_1 in ha case holds a mos 2^p ver ices for any $n \geq 5$. Hence, ha case is impossible.

- $3 \leq b \leq \frac{n-1}{2}$. In ha case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n+1) \rceil + 1$, which is false for any n.

Finally, if here is a cu -edge AB in a MGG_n where n is odd ($n \geq 5$), hen i is a penden edge ; moreover, $AB = \{1, g_n\}$.

Corollary 1 *In a MGG_n ($n \geq 4$) :*

- *there is no (1,2)-type edge ;*
- *two vertices of degree 1 cannot be adjacent to the same vertex.*

5.2 Vertices of degree 2 in a MGG_n

Theorem 2 *In a MGG_n :*

1. *For all n even ($n \geq 6$), if there is a vertex A of degree 2 with incident edges AB and AC, then 1 and g_n valuate AB or AC, e.g. $\{1, g_n\} \subseteq AB \cup AC$.*
2. *For all n odd ($n \geq 9$), if $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ and if there is a vertex A of degree 2 with incident edges AB and AC, then $\{1, g_n\} \subseteq AB \cup AC$.*

Proof : Le 's suppose firs ha 1 doesn' valua e AB nor AC. Then he MBT roo ed in A holds a mos $3 \times 2^{g_n - 3} + 1$ ver ices. If n is even, we know ha $n \geq 2^{g_n - 1} + 2$. Subsequen ly, he MBT roo ed in A doesn' hold enough ver ices. If n is odd, we reach he same conclusion for all $n \geq 3 \times 2^{g_n - 3} + 3$ (we recall ha $g_n = p + 1$ when n is odd). In case $n = 3 \times 2^{g_n - 3} + 1$, he MBT roo ed in A holds exac ly he necessary number of ver ices, bu he RBT doesn' for any $n \geq 9$.

Analogously, we prove ha g_n valua es ei her AB or AC hanks o he RBT roo ed in A.

5.3 Vertices of degree d in a MGG_n

Theorem 3 *For all n even such that $(2^d - 1) \times 2^{p-d} \leq n \leq 2^p$ with $d \leq p - 2$, there is no vertex of degree less or equal to d in a MGG_n.*

Proof : Firs , we recall ha $g_n = p$ for such values of n. Suppose we have a ver ex A of degree $d_1 \leq d$. Then he MBT roo ed in A holds $(2^{d_1} - 1) \times 2^{g_n - d_1} + 1$ ver ices, which is only possible for $d_1 = d$ and $n = (2^d - 1) \times 2^{g_n - d}$. In ha case, hough, we show ha necessarily ime uni s $1, 2 \ldots d$ valua e he d edges inciden o A. Analogously for he RBT roo ed in A, ime uni s $g_n, g_n - 1 \ldots g_n - d + 1$ need o valua e hose edges oo. If we combine hose wo cons rain s, and as we supposed $g_n \geq d + 2$, we ge a MBT roo ed in A wi h wo less ver ices han needed. Therefore, here canno be any ver ex of degree less or equal o d in a MGG_n.

5.4 (1,3)-type edges in a MGG_n

Theorem 4 *For all n odd ($n \geq 13$), if $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$, there is no (1,3)-type edge in a MGG_n.*

Proof : Le us suppose we have a (1,3)- ype edge AB, where A is he ver ex of degree 1 and B_1 and B_2 he wo o her ver ices adjacen o B. Theorem 1 yields ha 1 and g_n valua e AB. Consequen ly, he MBT roo ed in A holds $3 \times 2^{g_n-3} + 2$ ver ices. In he case $n = 3 \times 2^{p-2} + 1$, however, some cons rain s on he valua ions are given hanks o he MBT and RBT roo ed in A (e.g. $2 \in BB_i$, $3 \in BB_j$ ($i \neq j$) and $(g_n - 1) \in BB_k$, $(g_n - 2) \in BB_l$ ($k \neq l$)). However, he MBT roo ed in A according o hese cons rain s holds a mos $3 \times 2^{p-2} - 4$ ver ices.

5.5 Adjacency of vertices of degree 2 in a MGG

Theorem 5 *For all n even ($n \geq 8$ and $n \neq 2^{g_n-1} + 2$), two vertices of degree 2 can't be adjacent in a MGG_n.*

Proof : Le 's suppose we have wo ver ices of degree 2 adjacen , A and B. Theorem 2 yields ha 1 and g_n valua e he edges adjacen o A and B, which leads o four possible valua ion schemes. As previously, we de ermine some more cons rain s for each of hose schemes such ha he MBT roo ed in A doesn' hold enough ver ices.

Theorem 6 *For all n odd, if $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, a vertex of degree 2 is adjacent to at least one vertex of degree 3 or more in a MGG_n.*

Proof : Le 's suppose we have hree ver ices of degree 2, A, B and C, such ha A has B and C as neighbours. Theorem 2 yields ha 1 and g_n valua e he edges adjacen o each of hose ver ices. In roducing firs he valua ion 1, we ge a MBT roo ed in A wi h $3 \times 2^{g_n-3} + 2$ ver ices. Moreover, in he case $n = 3 \times 2^{g_n-3} + 1$, we in roduce he valua ion g_n and ge o a MBT roo ed in A wi h $3 \times 2^{g_n-3} - 4$ ver ices.

6 Upper bounds for G_n

6.1 Immediate property

Theorem 7 *For all n even, $G(n) \leq \frac{n \times g_n}{2}$; for all n odd, $G(n) \leq \frac{(n-1) \times g_n}{2}$.*

Proof : Suppose firs n is even. In order for an edge o be valua ed by a leas one ime uni , each ver ex is a worse of degree g_n. In ha case, he corresponding MGG_n is g_n-regular and has $\frac{n \times g_n}{2}$ edges.

If n is odd, he proof is similar excep ha we know ha , a each ime uni , a leas one ver ex doesn' communica e wi h any o her. So ha $2 \times G(n) \leq n \times g_n - g_n$, hence he resul .

6.2 Gossip graphs with $2n$ vertices

6.2.1 Knödel graphs

Definition 1 *The* Knödel graph *[FP94] on $n \geq 2$ vertices (n even) and of maximum degree $\Delta \geq 1$ is denoted $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the couples (i, j) with i=1,2 and $0 \leq j \leq \frac{n}{2} - 1$. For every j, $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, j + 2^k - 1 \bmod \frac{n}{2})$, for $k = 0, \ldots, \Delta - 1$.*

For $0 \leq k \leq \Delta - 1$, an edge of $W_{\Delta,n}$ which connec s a ver ex $(1, j)$ o he ver ex $(2, j + 2^k - 1 \bmod \frac{n}{2})$ is said o be *in dimension k*.

Theorem 8 *For all n even :*

- *if $2^{p+1} + 2 \leq n \leq 3 \times 2^p - 4$, then $G(n) \leq \frac{n \times (g_n - 2)}{2}$;*
- *if $3 \times 2^p - 2 \leq n \leq 2^{p+1} - 2$, then $G(n) \leq \frac{n \times (g_n - 1)}{2}$.*

Proof : We recall firs ha in ha case, $g_n = p + 2$. If we build he Knödel graph $W_{p,n}$ wi h $n \leq 3 \times 2^p - 4$, we can find a gossip scheme showing ha his graph is a gossip graph. Thus he firs upper bound for $G(n)$. If $n > 3 \times 2^p - 4$, hen we show ha $W_{p+1,n}$ is a gossip graph, which gives he second upper bound. We refer o [Fer97] for he de ailed proof of his heorem.

6.2.2 Definitions

- Le 's suppose we have wo graphs G_1 and G_2, bo h of order n. We call *composition of two graphs G_1 and G_2* any graph G of order $2n$ buil by adding any perfec ma ching be ween G_1 and G_2.
- A *self-composition* of a graph G is a composi ion of G by i self. For ins ance, he care esian produc by K_2 of any graph G is a self-composi ion of G.

6.2.3 Composition of two MGG_n

Theorem 9 *For all n even, $G(2n) \leq 2 \times G(n) + n$.*

Proof : If we self-compose a MGG_n, we ge a graph wi h $2 \times G(n) + n$ edges, able o gossip in $g_n + 1 = g_{2n}$, as shown in Figure 2. We hen ge a gossip graph wi h $2 \times G(n) + n$ edges.

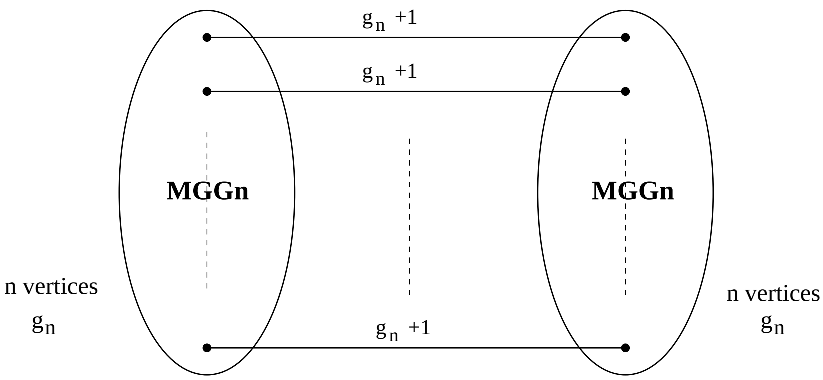

Figure 2: A gossip graph wi h $2 \times n$ ver ices

Corollary 2 *If a graph G is a MGG of order $2^p - 2$ ($p \geq 5$), then any self-composition of G is a MGG of order $2^{p+1} - 4$.*

Proof : Labahn [Lab93] showed ha $G(2^p - 2) = (p - 1) \times (2^{p-1} - 1)$ and ha $G(2^{p+1} - 4) = p \times (2^p - 2)$. Then he self-composi ion of G is a gossip graph, following he scheme shown in Figure 2, wi h a minimum number of edges.

6.3 Gossip graphs with $2n_1 + n_2$ vertices (n_i even)

Theorem 10 *For all (n_1, n_2) such that n_i even, $n_i \leq 2^{p-1}$ ($i \in \{1, 2\}$), $2 \times n_1 \geq n_2$ and $2 \times n_1 + n_2 > 2^p$, $G(2 \times n_1 + n_2) \leq 2 \times G(n_1) + G(n_2) + n_1 + \frac{n_2}{2}$.*

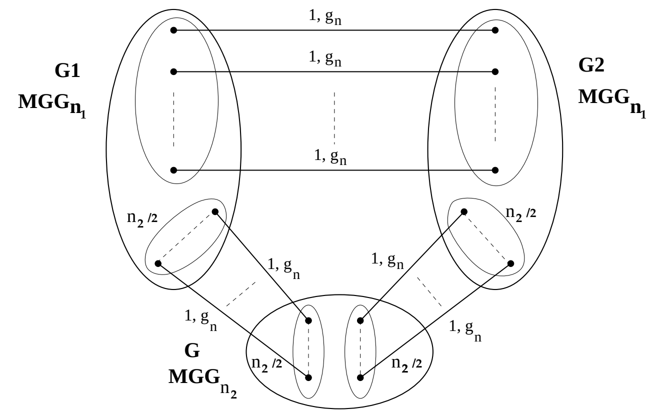

Figure 3: A gossip graph wi h $2 \times n_1 + n_2$ ver ices

Proof : The cons rain s on n_1 and n_2 ensure us ha g_n and g_{n_i} differ from 2 ime uni s a leas for $i \in \{1, 2\}$. Le 's now ake 2 MGG_{n_1} and a MGG_{n_2}. If we compose subgraphs of hose $MGGs$ as shown in Figure 3, and if we valua e he edges added for hose composi ions by 1 and g_n while he hree $MGGs$ gossip be ween 2 and $(g_n - 1)$, we ge a gossip graph wi h $2 \times G(n_1) + G(n_2) + n_1 + {}^{n_2}$

he remaining ver ices when composing $G_1 - \{U\}$ by $G_2 - \{V, W\}$ and valua ing he edges added for he composi ion wi h $(g_n + 2)$.

6.5 (1,q)-type edges in a MGG_n

Theorem 12 *For all n odd such that $2^p + 1 \leq n \leq 2^{p+1} - 5$, we partition the possible values of n in four classes :*

- *If $n = 4 \times m - 1$:*
  - *if $n \leq 3 \times 2^{p-1} - 9$, then $G(n) \leq m \times p - 1$;*
  - *if $n \geq 3 \times 2^{p-1} - 5$, then $G(n) \leq m \times (p + 1) - 1$.*
- *If $n = 4 \times m + 1$:*
  - *if $n \leq 3 \times 2^{p-1} - 11$, then $G(n) \leq (m + 1) \times p - 3$;*
  - *if $n \geq 3 \times 2^{p-1} - 7$, then $G(n) \leq (m + 1) \times (p + 1) - 3$.*

Proof : We recall ha , in ha case $g_n = p + 2$. The aim is o ob ain a gossip graph hanks o a cons ruc ion based on Knödel graphs, o which we add some penden edges. If $n = 4 \times m - 1$, ake a Knödel graph of order $2m$, and add $(2m - 1)$ penden edges o ge a graph of order n. O herwise, ha is $n = 4 \times m + 1$, ake a Knödel graph wi h $2(m + 1)$ ver ices and add $(2m - 1)$ penden edges. We know ha here is a gossip in he Knödel graph in $p = g_n - 2$ ime uni s (cf. Theorem 8). If we valua e he penden edges wi h ime uni s 1 and $(p+2)$ while he Knödel graph gossips be ween ime uni s 2 and $(p + 1)$, we see ha his graph is a gossip graph. However, he number of edges of such a graph depends on he value of $2m$ - ha is of n -, following Theorem 8. We refer o [Fer97] for a de ailed proof of he heorem.

Theorem 13 *For all n even such that $2^{p-1} < n \leq 2^p$, and for all k odd such that $2^p - n < k < n$, $G(n + k) \leq G(n) + k$.*

Proof : S ar ing from G, a MGG_n, we add k (1,q)- ype edges. Valua e hose penden edges wi h 1 and $(g_n + 1)$ while G gossips be ween 2 and g_n leads o a gossip graph wi h $G(n) + k$ edges.

Corollary 3 *For all n odd, $2^p + 1 \leq n \leq 2^{p+1} - 1$, $G(n) \leq n + (p - 2) \times 2^{p-1}$.*

Proof : As previously, we add $k = n - 2^p$ penden edges o he hypercube Q_p, and valua e he edges he same way in order o ge a gossip graph.

Theorem 14 *If G is a MGG_n with $2r$ vertices of degree 1 (n odd) such that $2^p + 2r + 1 \leq n \leq 2^{p+1} - 1$, then $G(n - 2r) \leq G(n) - 2r$.*

Proof : As, for hese values of n, $g_{n-2r} = g_n$, we ge a gossip graph wi h $G(n) - 2r$ edges and $(n - 2r)$ ver ices, removing from he MGG_n $2r$ ver ices of degree 1. We recall his removal can be done hanks o Theorem 1.

7 Lower bounds for G_n

7.1 Immediate property

Theorem 15 *For all $n \geq 4$, $G(n) \geq n$.*

Proof : As all $MGGs$ are connec ed, we know ha $G(n) \geq n - 1$ for all n. Moreover, Proper y 4 yields $g_n \geq 2 \times \lceil \log_2 n \rceil - 1$ in a ree for all n. As $g_n = \lceil \log_2 n \rceil$ for n even and $g_n = \lceil \log_2 n \rceil + 1$ for n odd, we show ha rees can' be $MGGs$ for any $n \geq 4$. Consequen ly, $G(n) \neq n - 1$.

7.2 Lower bounds for $G(2m)$

Theorem 16 *For all n even such that $2^{p-1}+2 \leq n \leq 3 \times 2^{p-2}-2$ and $p \geq 4$, $G(n) \geq \lceil \frac{5n}{4} \rceil$.*

Proof : The comple e proof is in [Fer97]. As seen in Theorems 1 and 2, we know ha here is no ver ex of degree 1, and ha if here is a ver ex of degree 2, 1 and g_n valua e i s inciden edges. Moreover, Theorem 5 yields ha here can' be wo adjacen ver ices of degree 2, excep for $n = 2^p + 2$. We hen show in Lemma 1 ha if we have s ver ices of degree a leas 3, here will be a mos s ver ices of degree 2, and s andard calcula ions yield he asser ed formula.

Theorem 17 *For all n even such that $2^p - 2^d + 2 \leq n \leq 2^p$ with $p \geq d+2 \geq 5$, $G(n) \geq \lceil \frac{n(2p-2d+3)}{4} \rceil$.*

Proof : The comple e proof is in [Fer97]. Theorem 3 yields ha for such values of n here is no ver ex of degree less or equal o $(p-d)$. Hence, he minimum degree of any ver ex will be $(p-d+1)$ in a MGG_n. We hen show in Lemma 3 ha if we have s ver ices of degree $(p-d+1)$ in a MGG_n, here are a leas s ver ices of degree $(p-d+2)$ or more. S andard calcula ions hen yield he asser ed formula.

Theorem 18 *For all n even such that $2^p - 2^d \leq n \leq 2^p$ with $p \geq d+2$, $G(n) \geq \frac{(p-d+1)\times n}{2}$.*

Remark : No e ha he lower bound for $G(n)$ given in he previous heorem (Theorem 17) is be er han his one. However, his bound is available for $n = 2^p - 2^d$, which is no he case in Theorem 17.

Proof : As seen in Theorem 3, here canno be any ver ex of degree less or equal o $(p-d)$ in a MGG_n for such values of n ; hence he resul .

Corollary 4 *For all (p,q) such that $q \geq 2$ and $p \geq q+2$, $(p-q+1)\times(2^{p-1}-2^{q-1}) \leq G(2^p-2^q) \leq (p-1)\times(2^{p-1}-2^{q-1})$.*

Proof : This is done by induc ion on of degree 2. So ha here is a limi ed number of ver ices of degree 2. Ac ually, we can show ha if we have s ver ices of degree a leas 3 (ver ices inciden o a $(1, q)$- ype edge excluded) in a MGG_n, hen here are a mos $2s$ ver ices of degree 2. S andard calcula ions hen yield he asser ed formula.

Theorem 21 *For all* $n = 2^p - 1$, $G(n) \geq \lceil \frac{6n}{5} \rceil$.

Proof : In ha case, we ge some more cons rain s han previously. Consequen ly, we show ha if here are s ver ices of degree a leas 3 (ver ices inciden o $(1, q)$- ype edges excluded) in a MGG_n, hen here are a mos $\frac{3s}{2}$ ver ices of degree 2 ([Fer97]). This provides immedia ely he asser ed formula.

8 Some particular cases

8.1 Tights bounds for $G(13)$

Theorem 22 $16 \leq G(13) \leq 17$.

Proof : Corollary 3 and Theorem 20 give us respec ively he lower and upper bound for $G(13)$.

Corollary 5 *If there are at least six* $(1, 4)$*-type edges in a* MGG_{13}*, then* $G(13) = 17$.

Proof : Proof of Theorem 20 yields ha for $n = 13$, $G(n) \geq \frac{7n+r}{6}$, where r is he number of $(1, q)$- ype edges. Consequen ly, if $r \geq 6$ hen $G(13) \geq 17$. Theorem 22 hen yields $G(13) = 17$. Moreover, he only $(1, q)$- ype edges ha can exis are $(1, 4)$- ype ones because :

- $(1, 3)$- ype edges canno exis for $n = 13$ (cf. Theorem 4), herefore $q \geq 4$;
- As $g_{13} = 5$ and a $(1, q)$- ype edge is valua ed by wo ime uni s (namely 1 and g_n, cf. Theorem 1), $q \leq 4$.

8.2 Determination of $G(15)$

Theorem 23 $G(15) = 19$.

Proof : Theorem 21 and Corollary 3 give us respec ively he lower and upper bound : $18 \leq G(15) \leq 19$. Now le 's suppose $G(15) = 18$. As seen in proof of Theorem 21, in ha case we have 6 ver ices of degree 3 and 9 ver ices of degree 2. Bu , hanks o ools provided by he proof of Theorem 21, we show ha in ha case no MGG_{15} can be buil wi h such ver ices. Hence $G(15) \neq 18$. Thanks o Corollary 3, we know ha he hypercube H_3 o which 7 penden edges are added is a MGG on 15 ver ices.

8.3 Determination of $G(24)$

Theorem 24 $G(24) = 36$.

Proof : As seen in Theorem 18, we know ha $G(24) \geq \frac{3 \times 24}{2}$, ha is $G(24) \geq 36$. Moreover, in heir s udy of Minimum Linear Gossip Graphs, Fraigniaud and Pe ers [FP94] gave a Minimum Linear Gossip Graph wi h 24 ver ices and $G_{1,1}(24) = 36$ edges, where $G_{\beta,\tau}$ is he minimum number of edges of a graph ha achieves gossiping in $g_{\beta,\tau}(n) = \lceil \log_2 n \rceil \beta + (n-1)\tau$ ime uni s. As $G_{1,1}(n) \geq G(n)$ for any n, $G(24) \leq 36$, and herefore $G(24) = 36$. We refer o [FP94] for an example of MGG_{24}.

8.4 Determination of $G(28)$

Theorem 25 $G(28) = 56$.

Proof: Firs no ice ha Labahn [Lab93] gave he value of $G(n)$ for $n = 2^p - 4$, bu ha his only holds for $p \geq 6$. We know ha $G(28) \geq 56$, hanks o Theorem 3, where $d = 4$. Moreover, he cons ruc ion given in Theorem 9 o ge an upper bound for $G(2n)$ yields $G(2n) \leq 2 \times G(n) + n$. As $G(14) = 21$ [Lab93], we ge $G(28) \leq 56$. Hence $G(28) = 56$, and he self-composi ion of a MGG_{14} is a MGG on 28 ver ices.

8.5 Bounds for $G(n)$ for $1 \leq n \leq 32$

To he au hor's knowledge, no bounds on $G(n)$ are available in he li era ure when $G(n)$ is no precisely known. Therefore, for $n = 13$, $n = 15$ and $17 \leq n \leq 32$, he resul s shown below are believed o be he bes known so far. Moreover, he resul s prin ed in bold charac ers indica e new exac resul s for $G(n)$, while he as erisk indica es op imali y.
These bounds are given hanks o he differen heorems and corollaries presen ed above, excep for $G(20) \geq 26$ and $G(22) \geq 31$, which are deduced from [MS94] (where we have $B(20) = 26$ and $B(22) = 31$) and Proper y 1.

n	Lower	Upper	n	Lower	Upper	n	Lower	Upper	n	Lower	Upper
1	0	0*	9	9	9*	17	18	20	25	30	32
2	1	1*	10	13	13*	18	23	27	26	46	52
3	2	2*	11	11	11*	19	20	22	27	32	34
4	4	4*	12	18	18*	20	26	30	28	**56**	**56***
5	5	5*	13	16	17	21	22	27	29	34	45
6	6	6*	14	21	21*	22	31	41	30	60	60*
7	7	7*	15	**19**	**19***	23	24	29	31	38	47
8	12	12*	16	32	32*	24	**36**	**36***	32	80	80*

Figure 5: Bounds for $G(n)$ ($1 \leq n \leq 32$)

8.6 Sum-up of the results for n even

8.6.1 Lower bounds for $G(n)$

n	$2^{p-1}+2\ldots3\times2^{p-2}-2$	$3\times2^{p-2}$	$3\times2^{p-2}+2\ldots2^p-2^d-2$	2^p-2d	$2^p-2^d+2\ldots2^p$
$G(n)$	$\lceil\frac{5n}{4}\rceil$	$\frac{3n}{2}$	$\lceil\frac{7n}{4}\rceil$	$\frac{n(p-d+1)}{2}$	$\lceil\frac{n(2p-2d+3)}{4}\rceil$

8.6.2 Upper bounds for $G(n)$

n	$2^{p-1}+2\ldots3\times2^{p-2}-4$	$3\times2^{p-2}-2\ldots2^p-2$	2^p
$G(n)$	$\frac{n(p-2)}{2}$	$\frac{n(p-1)}{2}$	$\frac{np}{2}$

8.7 Sum-up of the results for n odd

8.7.1 Lower bounds for $G(n)$

n	$2^{p-1}+1\ldots3\times2^{p-2}-1$	$3\times2^{p-2}+1\ldots2^p-3$	2^p-1
$G(n)$	$n+1$	$\lceil\frac{7n}{6}\rceil$	$\lceil\frac{6n}{5}\rceil$

8.7.2 Upper bounds for $G(n)$

n	$n = 4 \times m - 1$		$n = 4 \times m + 1$	
	$2^{p-1} + 1 \dots 3 \times 2^{p-2} - 9$	$3 \times 2^{p-2} - 5 \dots 2^{p} - 1$	$2^{p-1} + 1 \dots 3 \times 2^{p-2} - 11$	$3 \times 2^{p-2} - 7 \dots 2^{p} - 1$
$G(n)$	$m(p-1) - 1$	$mp - 1$	$(m+1)(p-1) - 3$	$(m+1)p - 3$

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9 Annex : complete proof of theorems

9.1 Proof of Theorem 1, page 4

We are going o give here he comple e proof of Theorem 1. Le 's suppose we have a cu -edge AB dividing a MGG_n in G_1 and G_2, as shown in Figure 6. Le 's suppose oo ha $AB = \{t_1 \ldots t_p\}$, where $1 \leq t_1 < t_2 \ldots < t_p \leq g_n$.

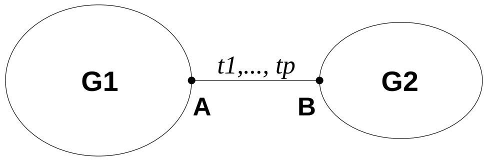


Figure 6: A cu -edge AB in a MGG_n

Le t_i be he smalles valua ion on AB such ha A knows he informa ion of each ver ex of G_1 in ime $(t_i - 1)$ a mos . Analogously, le t_j be he smalles valua ion on AB such ha B knows all he informa ion of G_2 in ime $(t_j - 1)$ a mos . We hen ge he four following inequali ies :

- $t_i - 1 \geq \lceil \log_2 (|V(G_1)|) \rceil$, as here has o be a broadcas from G_1 o A in a mos $(t_i - 1)$ ime uni s. Ye , we know ha he minimum ime o broadcas in he graph G_1 is $\lceil \log_2 (|V(G_1)|) \rceil$.
- Analogously, $t_j - 1 \geq \lceil \log_2 (|V(G_2)|) \rceil$, considering ha here mus be a broadcas from G_2 o B in $(t_j - 1)$ ime uni s a mos .
- $g_n - t_i \geq \lceil \log_2 (|V(G_2)|) \rceil$, because af er ime uni t_i, B knows all he informa ion of G_1 and needs o broadcas i hrough G_2 be ween ime uni s $(t_i + 1)$ and g_n.
- $g_n - t_j \geq \lceil \log_2 (|V(G_1)|) \rceil$. Analogously, if we consider he ver ex A, i has $(g_n - t_j)$ ime uni s o ransmi he informa ion hrough G_1.

If we add for ins ance he firs inequali y o he hird, we ge :

$$g_n - 1 \geq \lceil \log_2 a \rceil + \lceil \log_2 b \rceil \ \textbf{(I1)}.$$

where $a = |V(G_1)|$ and $b = |V(G_2)|$, and $a + b = n$. W.l.o.g., le us consider ha $a \geq \lceil \frac{n}{2} \rceil$.

Now suppose n even, and le us dis inguish wo cases :

- $b = 1$. In ha case, $a = n - 1$ and **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n-1) \rceil + 1$, which is false for any $n \geq 4$.
- $2 \leq b \leq \frac{n}{2}$. In ha case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 n \rceil + 1$, as we supposed $a \geq \frac{n}{2}$. However, his is false for any n.

Hence, if n is even $(n \geq 4)$, here is no cu -edge in a MGG.

Le n be odd, and a and b as above. Le us dis inguish hree cases ;

- $b = 1$, ha is BA is a penden edge. Then **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n-1) \rceil$, which is rue for any n. In ha case, he four inequali ies above lead o wo cases :
  1. If $n \neq 2^p + 1$, we ge $t_j \leq 1$ and $t_i \geq g_n$;
  2. If $n = 2^p + 1$, we ge $t_j \leq 2$ and $t_i \geq g_n - 1$, which leads o four possible valua ion schemes for he penden edge BA. Le us show ha only he case $\{t_j, t_i\} = \{1, g_n\}$ is possible.

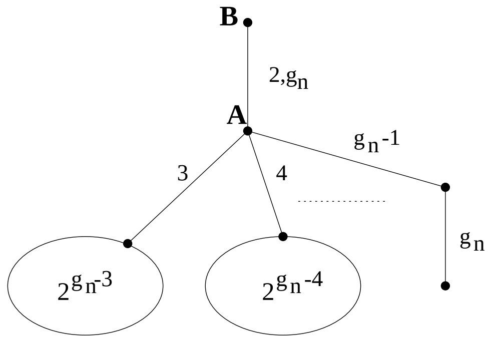


Figure 7: MBT roo ed in B wi h $BA = \{2, g_n\}$

- If $\{t_j, t_i\} = \{2, g_n\}$, he MBT roo ed in B, as shown in Figure 7, holds $|V(MBT)| = 1+1+2^{g_n-3}+2^{g_n-4}+\ldots+2$ ver ices, ha is $|V(MBT)| = 2^{g_n-2}$. As $n = 2^{g_n-2}+1$, we show ha his case is impossible.
- If $\{t_j, t_i\} = \{1, g_n -)|\}$ **a** $B)$ **a** **a** $G)$ **a** $(M))$ **a** $]\}$ **a** M **a** B **a** $M)$

- Finally, if $\{t_j, t_i\} = \{1, g_n\}$, he MBT as well as he RBT roo ed in B bo h hold $|V(MBT)| = 2^{g_n-1} - 1$ ver ices, which remains possible.

  Hence he MBTs and RBTs roo ed in B hold enough ver ices only in he case $BA = \{1, g_n\}$.

- $b = 2$. In ha case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n-2) \rceil + 1$, which is rue only for $n = 2^p + 1$. Le $V(G_1) = \{B, B_1\}$. Then BB_1 is a penden edge and, as seen above, we have $BB_1 = \{1, g_n\}$. Moreover, he firs four inequali ies yield $t_j = 2$ and $t_i = g_n - 1$. If we consider hen he MBT roo ed in B_1, as shown in Figure 10, we ge $|V(MBT)| = 2^{g_n-2}$ for any $g_n \geq 4$, ha is for any $n \geq 5$. As $n = 2^{g_n-2} + 1$, we show ha his case is impossible.

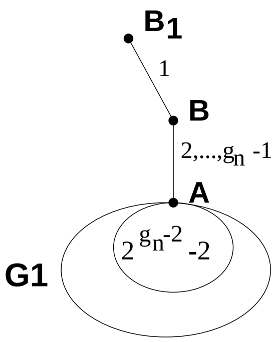


Figure 10: MBT roo ed in B_1 where $a = 2^{g_n-2} - 1$ and $b = 2$

- $3 \leq b \leq \frac{n-1}{2}$. In ha case, **(I1)** becomes $\lceil \log_2 n \rceil \geq \lceil \log_2 (n+1) \rceil + 1$, as we supposed $a \geq \frac{n+1}{2}$. However, his is false for any n.

Finally, if here is a cu -edge AB in a MGG_n where n is odd ($n \geq 5$), hen i is a penden edge ; moreover, $AB = \{1, g_n\}$.

9.2 Proof of Theorem 2, page 5

If 1 doesn' valua e AB nor AC, we ge he MBT roo ed in A shown in Figure 11, which holds $|V(MBT)| = 3 \times 2^{g_n-3} + 1$ ver ices.

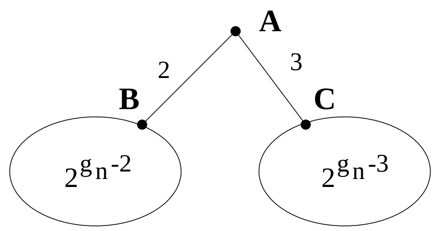


Figure 11: MBT roo ed in A ($1 \notin AB$, $1 \notin AC$)

Consequen ly, he only possible case is when $n = 3 \times 2^{g_n-3} + 1$. In ha case, as $|V(MBT)|$ has exac ly n ver ices, we need o have, w.l.o.g., $2 \in AB$ and $3 \in AC$. Then suppose $(g_n - t_1)$ and $(g_n - t_2)$ are he maximum values respec ively on AB and AC, wi h $t_1 \neq t_2$ and $t_i \geq 0$ ($i \in \{1, 2\}$). We ge he MBT roo ed in A shown in Figure 12.

This MBT holds :

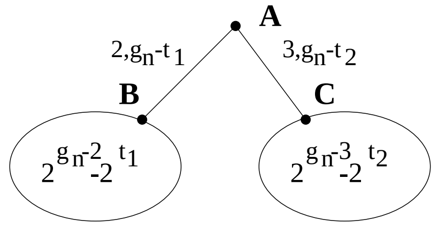


Figure 12: MBT roo ed in A

- $|V(MBT)| = 3 \times 2^{g_n - 3} + 1$ if $g_n - t_1 = 2$ and $g_n - t_2 = 3$;
- $|V(MBT)| = 3 \times 2^{g_n - 3} + 1 - 2^{t_2}$ if $g_n - t_1 = 2$ and $g_n - t_2 \neq 3$;
- $|V(MBT)| = 3 \times 2^{g_n - 3} + 1 - 2^{t_1}$ if $g_n - t_2 = 3$ and $g_n - t_1 \neq 2$;
- $|V(MBT)| = 3 \times 2^{g_n - 3} - 2^{t_1} - 2^{t_2} + 1$ o herwise.

Which shows he MBT roo ed in A holds enough ver ices only in he firs case. However, in his case we ge a RBT roo ed in A wi h 7 ver ices, as $(g_n - t_1)$ is he bigges valua ion on AB and $(g_n - t_2)$ he bigges valua ion on AC. As we supposed $n \geq 9$, 1 valua es necessarily AB or AC.

By a comple ely symme rical proof (looking a $RBTs$ ins ead of $MBTs$, and vice-versa), we prove ha g_n also valua es ei her AB or AC.

9.3 Proof of Theorem 3, page 5

Le A be a ver ex of degree e, where $1 \leq e \leq d$. Then he MBT roo ed in A, as shown in Figure 13, holds $|V(MBT)| = (2^e - 1) \times 2^{g_n - e} + 1$ ver ices.

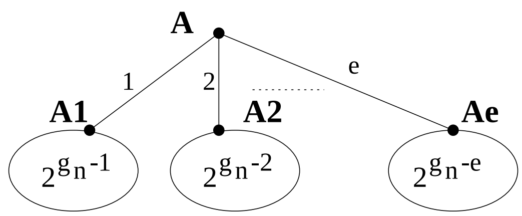


Figure 13: MBT roo ed in A of degree e

Clearly, his is only possible when $e = d$ and $n = (2^d - 1) \times 2^{p-d}$. In ha case, le 's suppose ha one of he edges AA_i $(1 \leq i \leq d)$ is such ha $i \notin AA_i$. The MBT roo ed in A, as shown in Figure 14, hen holds $|V(MBT)| = 2^{g_n} - 2^{g_n - d - 1} - 2^{g_n - i} + 1$ ver ices, which is s ric ly less han n for all i since $i \leq d \leq g_n - 2$.

Consequen ly, $i \in AA_i$ for any $i \in \{1, 2, \ldots d\}$. Symme rically, hanks o he RBT roo ed in A, we show ha he valua ions $g_n, g_n - 1, \ldots, g_n - d + 1$ valua e necessarily he edges inciden o A. As $g_n \geq d + 2$, ime uni s g_n and $(g_n - 1)$, a leas , are dis inc from $1, 2 \ldots d$. Hence, he number of ver ices of he MBT roo ed in A will remain he same wherever he valua ions g_n and $(g_n - 1)$ may be placed. Consequen ly, suppose w.l.o.g. ha $g_n \in AA_1$ and $(g_n - 1) \in AA_2$. Then he MBT roo ed in A, as shown in Figure 15, holds a mos $(2^d - 1) \times 2^{g_n - d} - 2$ ver ices, which is s ric ly less han n.

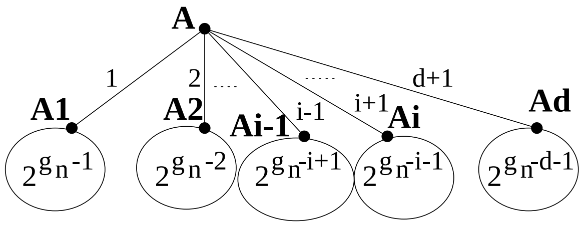


Figure 14: MBT roo ed in A where $i \notin AA_i$

Consequen ly, any MGG_n for hese values of n canno hold any ver ex of degree less or equal o d.

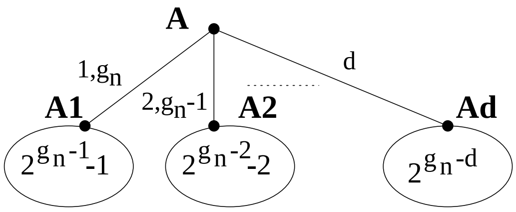


Figure 15: MBT roo ed in A

9.4 Proof of Theorem 4, page 6

Suppose AB is a $(1,3)$- ype edge. We already know ha $AB = \{1, g_n\}$ hanks o Theorem 1. We recall ha we supposed $n \geq 13$, which means $g_n \geq 5$. The MBT roo ed in A, as shown in Figure 16, has $|V(MBT)| = 3 \times 2^{g_n-3} + 2$ ver ices. Because n is odd, he only possible case is when $n = 3 \times 2^{g_n-3} + 1$.

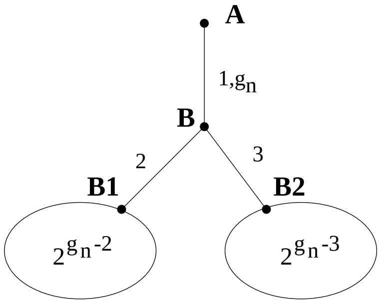


Figure 16: MBT roo ed in A where AB is a (1,3)- ype edge

In ha case, le 's suppose ha $2 \notin BB_1 \cup BB_2$. The MBT roo ed in A, as shown in Figure 17, has $|V(MBT)| = 3 \times 2^{g_n-4} + 2$ ver ices, which is s ric ly less han n for any $g_n \geq 4$.

Analogously, suppose $3 \notin BB_1 \cup BB_2$. Figure 18 yields ha he MBT roo ed in A holds $|V(MBT)| = 5 \times 2^{g_n-4} + 2$, which is also s ric ly less han n for any $g_n \geq 5$.

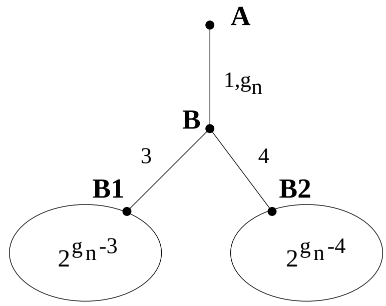


Figure 17: MBT roo ed in A supposing $2 \notin BB_i$

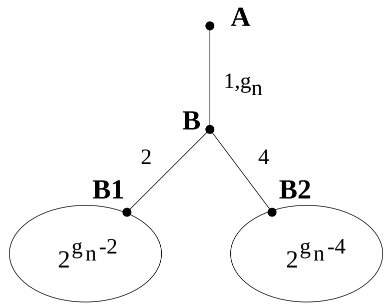


Figure 18: MBT roo ed in A supposing $3 \notin BB_i$

Symme rically, using he $RBTs$ roo ed in A in each case, we prove ha $(g_n - 1) \in BB_j$ and $(g_n - 2) \in BB_k$. We hen ge wo possible valua ion schemes, as shown in Figure 19.

We know ha $g_n \geq 5$. Le us dis inguish wo cases :

- If $g_n = 5$, hen he only possible valua ion scheme is he lef one in Figure 19 and $|V(MBT)| = 3 \times 2^{g_n - 3}$.
- If $g_n > 5$, hen he wo valua ion schemes lead o a MBT roo ed in A wi h $|V(MBT)| = 3 \times 2^{g_n-3} - 4$ ver ices.

In every case, we see ha none of he $MBTs$ roo ed in A holds enough ver ices : for hose values of n, here is no (1,3)- ype edge.

9.5 Proof of Theorem 5, page 6

Suppose we have wo ver ices of degree 2 adjacen , A and B, and A_1 (resp. B_1) is he o her ver ex adjacen o A (resp. B). Knowing ha 1 and g_n valua e he edges adjacen o A and B, we ge four possibili ies. Le us dis inguish hem :

1. Suppose $1 \in AB$. Le us deno e by p (resp. q) he smalles valua ion on AA_1 (resp. BB_1). As shown in Figure 20, he MBT roo ed in A has $|V(MBT)| = 2^{g_n-p} + 2^{g_n-q} + 2$ ver ices. We no ice ha $\forall\ (p, q)$, $|V(MBT)| \leq 2^{g_n-1} + 2$, which means ha he only possible case would be for $n = 2^{g_n-1} + 2$, and, in ha case, $p = q = 2$.

   Now, considering where he valua ion g_n could be, we ge wo cases.

   (a) If $g_n \in AB$, hen, reasoning symme rically, we can prove ha $(g_n - 1) \in AA_1$ and $(g_n - 1) \in BB_1$. Unless $g_n = 2$, he MBT roo ed in A, as shown in Figure 21, has

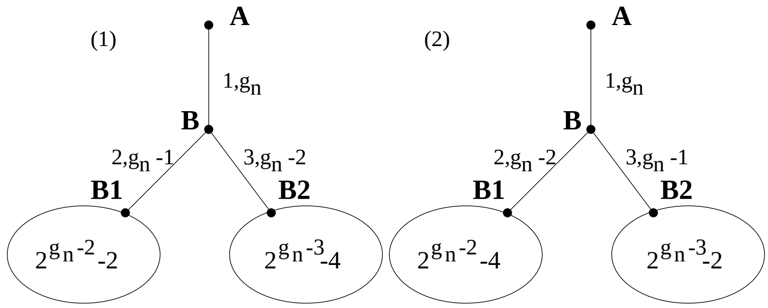


Figure 19: $MBTs$ roo ed in A where AB is a (1,3)- ype edge

8 -24.7455 Td (A1)Tj 89.4645 0 Td (B1)Tj /R443 6.8526 Tf -87.561 -17.1315 Td (2)Tj 7189591 2

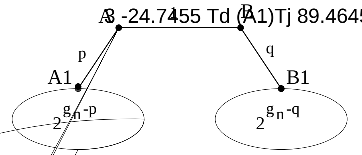


Figure 20: MBT roo ed in A

ices, which is s ric ly less han n. If $g_n = 2$, however, we
d $n \geq 8$, we don' have o consider his case.

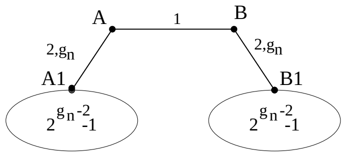


Figure 22: MBT roo ed in A

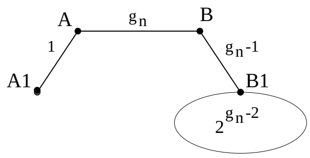


Figure 23: RBT roo ed in A, supposing $|AA_1| = 1$

- $|V(MBT_B)| = 2^{g_n-1} + 2^{g_n-p} - 2^{g_n-q} + 2$ ver ices ;

Consequen ly, n can a mos be equal o $2^{g_n-1} + 2$, and in ha case $p = q$. Following he hypo hesis ha $2^{g_n-1} + 2 \leq n \leq 2^{g_n}$, hen $n = 2^{g_n-1} + 2$. Le 's dis inguish wo cases :

i. If $p \neq g_n$, hen he MBT roo ed in A, as shown in Figure 26, holds $|V(MBT)| = 2^{g_n-1}$ ver ices, which is of course no enough ;

ii. If $p = q = g_n$, ha is $AA_1 = \{1, g_n\}$ and $BB_1 = \{1, g_n\}$, hen he $RBTs$ and $MBTs$ roo ed in A and B all hold $|V(MBT)| = n$ ver ices, which remains possible.

Then here canno be any wo ver ices of degree 2 adjacen for any n even wi h $n \geq 8$ and $n \neq 2^{g_n-1} + 2$; if $n = 2^{g_n-1} + 2$, hen necessarily $AA_1 = \{1, g_n\}$ and $BB_1 = \{1, g_n\}$.

9.6 Proof of Theorem 6, page 6

Le 's suppose we have hree ver ices of degree 2, A, B, C such ha he neighborhood of A is $\{B, C\}$. We know, hanks o Theorem 2 ha 1 and g_n valua e he edges adjacen o hese ver ices. W.l.o.g., hanks o he symme ry in A, le 's suppose $1 \in AB$. Then he MBT roo ed in A, as shown in Figure 27 holds $|V(MBT)| = 3 \times 2^{g_n-3} + 3$ ver ices, which remains possible for $n = 3 \times 2^{g_n-3} + 3$ and $n = 3 \times 2^{g_n-3} + 1$.

In bo h cases, le 's show ha 2 valua es AC. If $2 \notin AC$, hen, as shown in Figure 28, he MBT roo ed in A has $|V(MBT)| = 5 \times 2^{g_n-4} + 3$ ver ices, and $|V(MBT)| < n$ for any $g_n \geq 6$.

If $g_n = 5$, however, we need o in roduce he valua ion g_n o ge he resul . If $g_n \in AB$, hen Theorem 2 implies ha $g_n \in CC_1$, as shown in Figure 29 (lef figure). Analogously, if $g_n \in AC$ hen $g_n \in BB_1$ oo. We recall ha in ha case $g_n = 5$ and ha we suppose $n \in \{13, 15\}$. Figure 29 shows ha , wherever he valua ion g_n is, we ge a MBT roo ed in A wi h 12 ver ices.

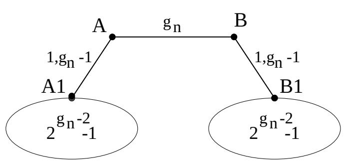


Figure 24: RBT roo ed in A

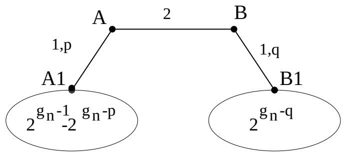


Figure 25: MBT roo ed in A

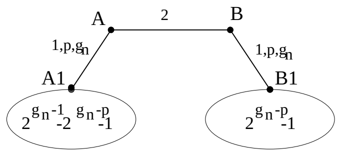


Figure 26: MBT roo ed in A where $p \neq g_n$

Then, we know ha $1 \in AB$ and $2 \in AC$. By a comple ely analogous proof, we show ha , similarly, $2 \in BB_1$ and $3 \in CC_1$. Moreover, by a symme rical proof (looking a $RBTs$ ins ead of $MBTs$, and vice-versa), we show ha g_n is ei her in AB or AC, and ha :

- If $g_n \in AB$, $(g_n - 1) \in AC$, $(g_n - 1) \in BB_1$ and $(g_n - 2) \in CC_1$;
- If $g_n \in AC$, $(g_n - 1) \in AB$, $(g_n - 1) \in CC_1$ and $(g_n - 2) \in BB_1$.

Moreover, Theorem 2 implies ha , since $1 \in AB$, hen $1 \in CC_1$, and ha :

- If $g_n \in AB$, hen $g_n \in CC_1$;
- If $g_n \in AC$, hen $g_n \in BB_1$.

The case $g_n \in AB$ refers o he lef figure in Figure 30, as he case $g_n \in AC$ refers o he righ one. In bo h cases, we show ha he MBT roo ed in A holds :

- $|V(MBT)| \leq 3 \times 2^{g_n - 3}$ ver ices for $g_n = 5$;
- $|V(MBT)| = 3 \times 2^{g_n - 3} - 4$ ver ices for $g_n > 5$.

We hen prove ha , in any case, here can' be a ver ex of degree 2 adjacen o wo ver ices of degree 2 for any n odd such ha $n \geq 3 \times 2^{g_n - 3} + 1$ wi h $g_n \geq 5$.

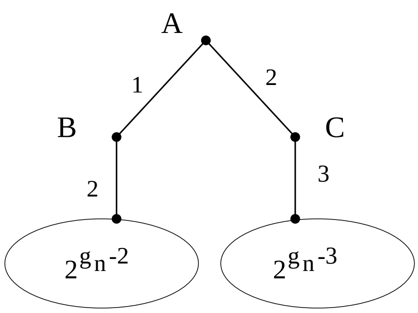


Figure 27: 3 adjacen ver ices of degre 2 : MBT roo ed in A

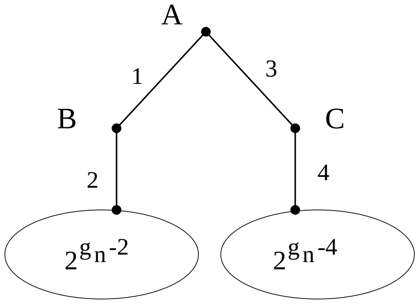


Figure 28: MBT roo ed in A, supposing $2 \notin AB$

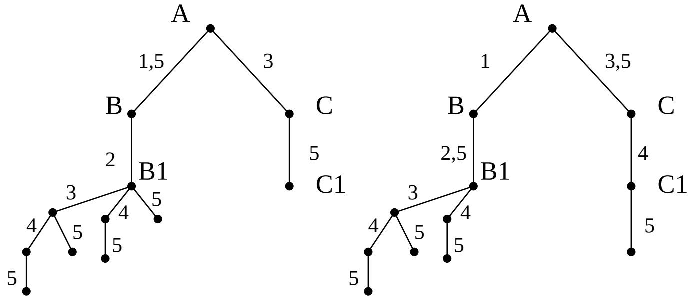


Figure 29: $MBTs$ roo ed in A where $g_n = 5$

9.7 Proof of Theorem 8, page 7

The aim is o show ha :

- For n even such ha $2^{p+1} + 2 \leq n \leq 3 \times 2^p - 4$, $W_{p,n}$ is a gossip graph ;
- For n even such ha $3 \times 2^p - 2 \leq n \leq 2^{p+2} - 2$, $W_{p+1,n}$ is a gossip graph.

9.7.1 $2^{p+1} \leq n \leq 3 \times 2^p - 4$

Le us prove ha $W_{p,n}$ is a gossip graph. Firs , we recall ha $g_n = p + 2$. Le he gossip scheme be he following :

- For i in $1 \dots p$, ver ices communica e hrough dimension $(i-1)$ during ime uni i ;
- Ver ices communica e hrough dimension 0 during ime uni $p+1$;
- Ver ices communica e hrough dimension $(p-1)$ during ime uni $p+2$.

Now le us show ha his gives a valid gossip scheme. I is easy o see ha af er a round $q \leq p$, a ver ex $(1, j_0)$ knows he pieces of informa ion of all he ver ices (i,j), $i = 1, 2$ and $j = j_0, j_0 + 1, \dots, j_0 + 2^{q-1} - 1 \bmod \frac{n}{2}$. Similarly, a ver ex $(2, j_0)$ knows he pieces of informa ion of all he ver ices (i,j), $i = 1, 2$ and $j = j_0, j_0 - 1, \dots, j_0 - 2^{q-1} + 1 \bmod \frac{n}{2}$ [FP94].

As every ver ex communica es hrough dimension 0 during ime uni $(p+1)$, a ver ex (i_0, j_0) knows all he pieces of informa ion of ver ices (i,j), $i = 1, 2$ and $j = j_0 - 2^{p-1} + 1 \bmod \frac{n}{2}, \dots, j_0 - 1, j_0, j_0 + 1, \dots, j_0 + 2^{p-1} - 1 \bmod \frac{n}{2}$. Finally, here will be a communica ion hrough dimension $(p-1)$. Le us consider a ver ex $(1, j_0)$ and a ver ex $(2, j_0 + 2^{p-1} - 1 \bmod \frac{n}{2})$. As seen above, $(1, j_0)$ knows he pieces of informa ion of (i,j), $i = 1, 2$ and $j = j_0 - 2^{p-1} + 1 \bmod \frac{n}{2}, \dots, j_0 -$

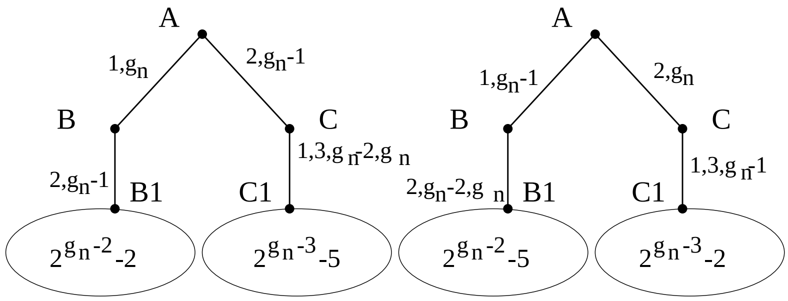


Figure 30: $MBTs$ roo ed in A

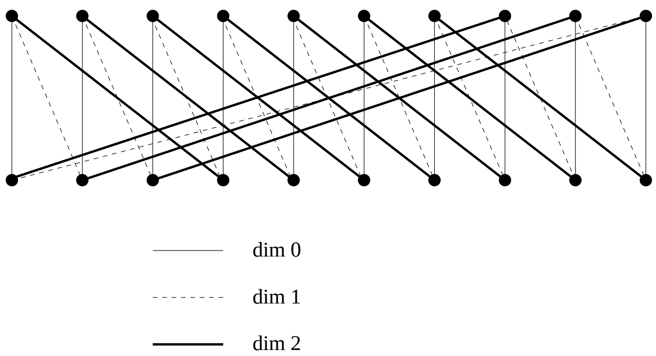


Figure 31: $W_{3,20}$ is a gossip graph

$1, j_0, j_0+1, \ldots, j_0+2^{p-1}-1 \bmod \frac{n}{2}$, and $(2, j_0+2^{p-1}-1 \bmod \frac{n}{2})$ knows he pieces of informa ion of (i,j), $i = 1,2$ and $j = j_0, j_0+1, \ldots, j_0+2^p-2 \bmod \frac{n}{2})$. During ime uni $p+2$, hose wo ver ices communica e. Thus, hey will know he pieces of informa ion of all he o her ver ices iff $j_0+2^p-2 \bmod \frac{n}{2}+1 \geq j_0-2^{p-1}+1 \bmod \frac{n}{2}$, ha is $n \leq 3 \times 2^p - 4$.

Consequen ly, if $2^{p+1}+2 \leq n \leq 3 \times 2^p - 4$, $W_{p,n}$ is a gossip graph and hus $G(n) \leq \frac{np}{2}$ where $p = g_n - 2$.

9.7.2 $3 \times 2^p - 2 \leq n \leq 2^{p+2} - 2$

Le us prove ha $W_{p+1,n}$ is a gossip graph. Le he gossip scheme be he following :

- For i in $1 \ldots p+1$, ver ices communica e hrough dimension $(i-1)$ during ime uni i ;
- Ver ices communica e hrough dimension 0 during ime uni $p+2$;

Similarly o he previous case, i is easy o see ha af er he $(p+1)$- h round, a ver ex $(1, j_0)$ knows all he pieces of informa ion of ver ices (i,j), $i = 1,2$ and $j = j_0, j_0+1, \ldots, j_0+2^p-1 \bmod \frac{n}{2}$. Similarly, a ver ex $(2, j_0)$ knows he pieces of informa ion of ver ices (i,j), $i = 1,2$ and $j = j_0, j_0-1, \ldots, j_0-2^p+1 \bmod \frac{n}{2}$.

As he las communica ion akes place hrough dimension 0, any ver ex (i_0, j_0) will know he pieces of informa ion of ver ices (i,j), $i = 1,2$ and $j = j_0-2^p+1 \bmod \frac{n}{2}, \ldots, j_0-1, j_0, j_0+1, \ldots, j_0+2^p-1 \bmod \frac{n}{2}$. Which means ha any ver ex (i_0, j_0) knows he pieces of informa ion of all he o her ver ices in he graph iff $j_0+2^p \bmod \frac{n}{2} \geq j_0-2^p+1 \bmod \frac{n}{2}$, ha is $n \leq 2^{p+2}-2$. Hence his gossip scheme is valid for all n no a power of wo and hus for $3 \times 2^p - 2 \leq n \leq 2^{p+2}-2$, $W_{p+1,n}$ is a gossip graph. Consequen ly, we ge $G(n) \leq \frac{n \times (p+1)}{2}$, where $p = g_n - 2$.

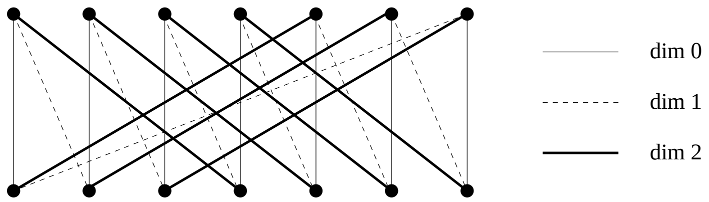


Figure 32: $W_{3,14}$ is a gossip graph

9.8 Proof of Theorem 12, page 9

Suppose we have $2^p + 1 \leq n \leq 2^{p+1} - 5$. Then $g_n = p + 2$. We will need o dis inguish wo cases depending on he value of n : $n = 4 \times m - 1$ and $n = 4 \times m + 1$.

9.8.1 $n = 4 \times m - 1$

Le us hen ake a Knödel graph on $2m$ ver ices. We know ha his graph can gossip in $p = g_n - 2$ ime uni s. Le us hen add $(2m - 1)$ penden edges. If we valua e hose edges wi h ime uni s 1 and $(p + 2)$ while he Knödel graph gossips be ween ime uni s 2 and $(p + 1)$, we ge a gossip graph. The number of edges depends on he Knödel graph considered, which depends i self on he number n of ver ices. As seen in Theorem 8, we have :

- If $2m \leq 3 \times 2^{p-2} - 4$, ha is $n \leq 3 \times 2^{p-1} - 9$, hen we can use he Knödel graph $W_{p-2,2m}$, which has $m \times (p - 2)$ edges. Thus in ha case $G(n) \leq m \times (p - 2) + (2m - 1)$, ha is $G(n) \leq m \times p - 1$.
- If $2m \geq 3 \times 2^{p-2} - 2$, ha is $n \geq 3 \times 2^{p-1} - 5$, hen we have o use he Knödel graph $W_{p-1,2m}$, which has $m \times (p - 1)$ edges. Thus $G(n) \leq m \times (p + 1) - 1$.

9.8.2 $n = 4 \times m + 1$

We use he same me hod as previously, excep ha we ake a Knödel graph wi h $2(m+1)$ ver ices and ha we add $(2m - 1)$ penden edges. Similarly, he maximum degree Δ of he considered Knödel graph will depend on he value of n.

- If $2(m + 1) \leq 3 \times 2^{p-2} - 4$, ha is $n \leq 3 \times 2^{p-1} - 11$, hen we can use he Knödel graph $W_{p-2,2(m+1)}$, which has $(m + 1) \times (p - 2)$ edges. Thus $G(n) \leq (m + 1) \times p - 3$.
- If $2(m + 1) \geq 3 \times 2^{p-2} - 2$, ha is $n \geq 3 \times 2^{p-1} - 7$, hen we need o use he Knödel graph $W_{p-1,2(m+1)}$, which has $(m+1) \times (p-1)$ edges. Consequen ly, $G(n) \leq (m+1) \times (p+1) - 3$.

9.9 Proof of Theorem 16, page 10

The aim here is o prove he following lemma :

Lemma 1 *In a MGG_n where n is even, if there are s vertices of degree at least 3, then there are at most s vertices of degree 2.*

Proof : We recall ha if n is even, here is no ver ex of degree 1. Moreover, concerning ver ices of degree 2, we know ha :

- For $n \neq 2^{g_n - 1} + 2$, if here is a ver ex A of degree 2 adjacen o B and C, hen B and C are of degree a leas 3 and $\{1, g_n\} \subseteq AB \cup AC$ (cf. Theorems 2 and 5).
- If $n = 2^{g_n - 1} + 2$, hen if a ver ex A is of degree 2 :

1. If one of i s neighbours B is of degree 2, hen le 's deno e by A_1 (resp.B_1) he ver ex adjacen o A (resp.B) which is no B (resp.A). Then $AA_1 = BB_1 = \{1, g_n\}$ (cf. Proof of Theorem 5) ;
2. If none of he neighbours B and C of A is of degree 2, hen $\{1, g_n\} \subseteq AB \cup AC$ (cf. Theorem 2).

Le us show now he lemma. We deno e by V_2 he se of ver ices of degree 2 in he graph and by V_{3+} he se of ver ices of degree a leas 3. Clearly $V = V_2 \cup V_{3+}$. Le Φ be he following func ion :
$\Phi : V_2 \longrightarrow V_{3+}$
$A \longmapsto B$ such ha $1 \in AB$.
Firs , Φ is a func ion, ha is each A in V_2 has an image in V_{3+}. Moreover, due o he 1-por model, Φ is injec ive, ha is if $A \neq A'$, hen $\Phi(A) \neq \Phi(A')$. Hence $|V_2| \leq |V_{3+}|$.

Le us now prove he heorem, ha is $G(n) \geq \lceil \frac{5n}{4} \rceil$. As here is no ver ex of degree 1 when n is even in a MGG_n, we will have ver ices of degree 2 and ver ices of degree 3 or more. Suppose we have s ver ices of degree a leas 3 in a MGG_n, ha is we have $(n - s)$ ver ices of degree 2. Lemma 1 yields ha $n - s \leq s$. Moreover, we know ha he sum of he degrees is wice he number of edges. Then, $2 \times G(n) \geq 3 \times s + 2 \times (n - s)$, ha is $2 \times G(n) \geq 2 \times n + s$. Thanks o he firs inequali y, we ge $G(n) \geq \frac{5n}{4}$. As $G(n)$ is an in eger, we ge he resul .

9.10 Proof of Theorem 17, page 10

Le n be an even number such ha $2^p - 2^d + 2 \leq n \leq 2^p$, wi h $p \geq d + 2 \geq 5$. Firs , as $n > 2^p - 2^d$, Theorem 3 yields ha here canno be any ver ex of degre less or equal o $(p - d)$. Consequen ly, he minimum degree will be $(p - d + 1)$. Now le us prove he following lemma :

Lemma 2 *In a MGG_n such that $2^p - 2^d + 2 \leq n \leq 2^p$ with n even and $p \geq d + 2 \geq 5$, if there is a vertex A of degre $(p - d + 1)$, then it is incident to an edge AB such that $1 \in AB$ and B is of degree at least $(p - d + 2)$.*

Proof : Suppose we have a ver ex A of degree $(p - d + 1)$. We recall ha , for such values of n, $g_n = p$. Firs le us show ha 1 (resp. 2) necessarily valua e one of he edges inciden o A. Suppose ha 1 does no valua e any of he edges inciden o A. Then, he MBT roo ed in A, as shown in Figure 33, holds $|V(MBT)| = 2^{p-1} - 2^{d-2} + 1$ ver ices, which is s ric ly less han n for any $p \geq d + 1$ (we recall ha we supposed $p \geq d + 2$). Hence 1 valua es one of he edges inciden o A. Le us deno e his edge AB.

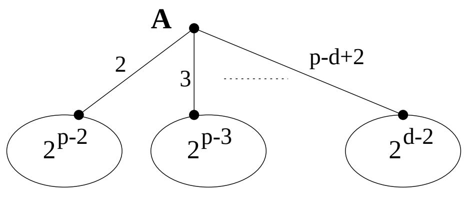


Figure 33: MBT roo ed in A

Similarly, suppose 2 does no valua e any of he edges inciden o A. Then he MBT roo ed in A, as shown in Figure 34, holds $|V(MBT)| = 3 \times 2^{p-2} - 2^{d-2} + 1$ ver ices, which is s ric ly less han n for any $p \geq d + 2$. Hence 2 valua es one of he edges inciden o A. Symme rically, we can show hanks o he RBTs roo ed in A ha p and $(p-1)$ necessarily valua e he edges inciden o A.

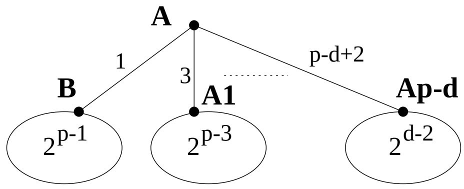


Figure 34: MBT roo ed in A

Now le us consider he ver ex B such ha $1 \in AB$, and le us prove ha B is of degree a leas $(p-d+2)$. Suppose B is of degree $(p-d+1)$. Then he MBT roo ed in A, as shown in Figure 35, holds exac ly $n = 2^p - 2^d + 2$ ver ices.

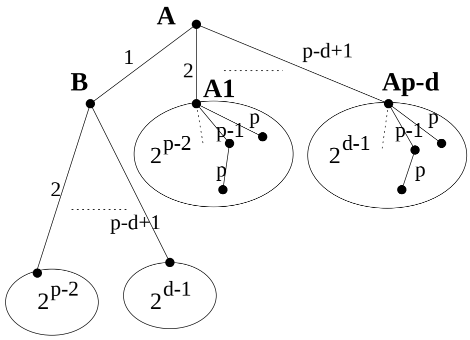


Figure 35: MBT roo ed in A

However, in ha case all of he ver ices $A_1 \ldots A_{p-d}$ are such ha $(p-1)$ and p valua e heir inciden edges, o herwise he MBT roo ed in A would no hold enough ver ices. Moreover, $d \geq 3$, which means ha p and $(p-1)$ are dis inc from any of he numbers $1, 2 \ldots, p-d+1$. Consequen ly, due o he 1-por model, he only edge inciden o A ha can be valua ed by p as well as $(p-1)$ is he edge AB. Bu bo h p and $(p-1)$ valua ing AB is poin less and leads o a RBT roo ed in A wi h s ric ly less han n ver ices.

We have hen shown ha if here is a ver ex A of degree $(p-d+1)$, hen i is adjacen o he ver ex B of degree a leas $(p-d+2)$ such ha $1 \in AB$. Now le us prove he following lemma :

Lemma 3 *In a MGG_n such that $2^p - 2^d + 2 \leq n \leq 2^p$ with n even and $p \geq d+2 \geq 5$, if there are s vertices of degree $(p-d+1)$, then there are at least s vertices of degree at least $(p-d+2)$.*

Proof : No e ha his proof is very close o he proof of Lemma 1. Le V_1 be he se of ver ices of degree $(p-d+1)$ in he graph and V_2 he se of ver ices of degree a leas $(p-d+2)$. Clearly $V = V_1 \cup V_2$. Le Φ be he following func ion :
$\Phi : V_1 \longrightarrow V_2$
$A \longmapsto B$ such ha $1 \in AB$.
Firs , Φ is a func ion, ha is each A in V_1 has an image in V_2 (Lemma 2). Moreover, due o he 1-por model, Φ is injec ive, ha is if $A \neq A'$, hen $\Phi(A) \neq \Phi(A')$. Hence $|V_1| \leq |V_2|$.

Finally, le us prove he heorem. We know ha in a MGG_n such ha $2^p - 2^d + 2 \leq n \leq 2^p$ wi h n even and $p \geq d+2 \geq 5$:

- The minimum degree is $(p-d+1)$;

- If here are s ver ices of degree $(p-d+1)$, hen here are a leas s ver ices of degree a leas $(p-d+2)$.

Hence le us suppose we have s ver ices of degree $(p-d+1)$ in a MGG_n. Then he $(n-s)$ remaining ver ices are of degree a leas $(p-d+2)$ and Lemma 3 yields $n-s \geq s$ **(1)**. Moreover, he sum of he degrees is wice he number of edges. Tha is : $2 \times G(n) \geq s \times (p-d+1) + (n-s) \times (p-d+2)$ **(2)**. Combining **(1)** and **(2)** leads o he asser ed formula, ha is : $G(n) \geq \lceil \frac{n \times (2p-2d+3)}{4} \rceil$.

9.11 Proof of Theorem 19, page 10

Suppose $G(n) = n$. Then, we know ha here can only be ver ices of degree 2 and/or ver ices con ribu ing o (1,3)- ype edges. We dis inguish wo cases :

- If here is no (1,3)- ype edge, hen he considered graph is C_n, cycle of leng h n. Consequen ly, $D(C_n) = \lfloor n/2 \rfloor$. As $\lfloor n/2 \rfloor > \lceil \log_2(n) \rceil + 1$ for any $n \geq 13$, C_n canno be MGG_n for any $n \geq 13$.
- If here is a leas one (1,3)- ype edge AB, hen we show, hanks o Figure 36, ha he MBT roo ed in A holds a mos $|V(MBT)| = 4 \times g_n - 8$ ver ices, wha ever he number of (1,3)- ype edges is. Ye , for any $n \geq 13$, $n > |V(MBT)|$, which shows ha case is impossible oo.

Consequen ly, $G(n) \neq n$ and $G(n) \geq n+1$.

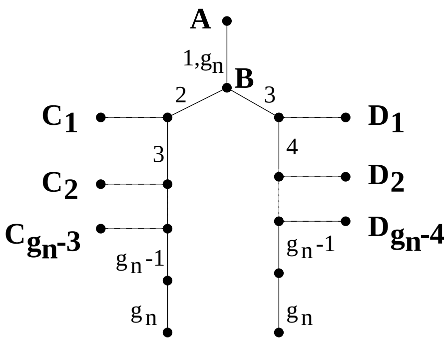


Figure 36: MBT roo ed in A where AB is a (1,3)- ype edge

9.12 Proof of Theorem 20, page 10

The cons ruc ion of he proof is he following : firs , we prove Proper ies 6 and 7 below. These proper ies are necessary o prove Lemma 4. Finally, Lemma 4 will allow us o prove he heorem.

Property 6 *For any n odd such that $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, if there are two adjacent vertices of degree 2, A and B, such that A is also adjacent to A_1 and B is also adjacent to B_1, then :*

- *If $1 \in AB$, then $2 \in AA_1$ and $2 \in BB_1$;*
- *If $g_n \in AB$, then $(g_n - 1) \in AA_1$ and $(g_n - 1) \in BB_1$.*
- *Moreover, in the case $n = 2^p - 1$, we cannot have $1 \in AB$ and $g_n \in AB$.*

Remark : We recall ha in ha case $g_n = p+1$.

Proof : Le 's consider wo adjacen ver ices of degree 2, A and B, as described above. Firs we recall ha Theorem 2 yields ha 1 and g_n valua e he edges adjacen o A and B, so ha if $1 \notin AB$, necessarily $1 \in AA_1$ and $1 \in BB_1$. No ice ha he same goes o g_n.

Suppose now we have $1 \in AB$. If $2 \notin AA_1$ and $2 \notin BB_1$, we ge a MBT roo ed in A wi h $|V(MBT)| = 2^{g_n-2}+2$ ver ices, as shown in he lef figure of Figure 37. However, $|V(MBT)| < n$ for any $g_n \geq 4$.

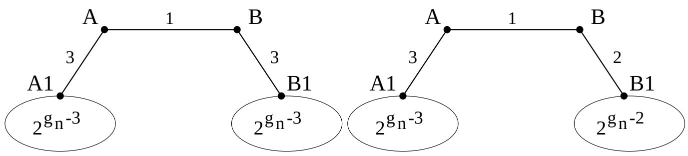
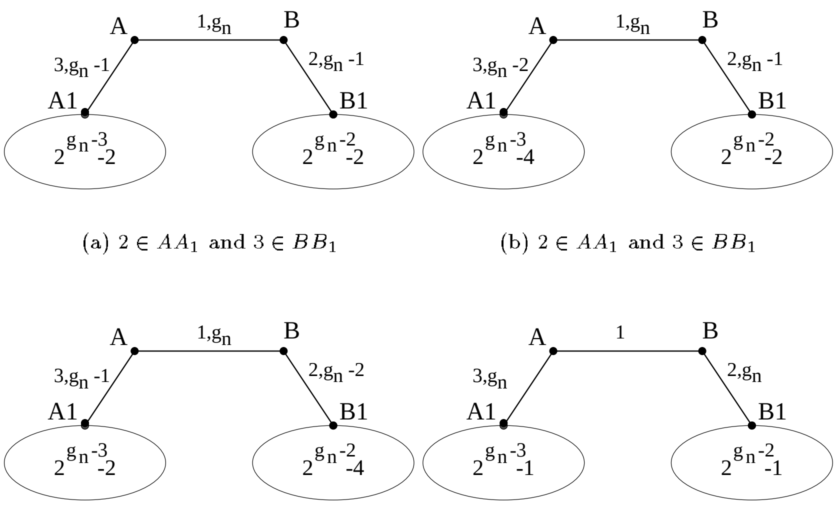


Figure 37: $MBTs$ roo ed in A

Suppose now, w.l.o.g., ha $2 \notin AA_1$ and $2 \in BB_1$. The MBT roo ed in A holds $|V(MBT)| = 3 \times 2^{g_n-3}+2$ ver ices, as shown in he righ figure of Figure 37. Now consider he valua ion g_n. If $g_n \in AB$, hen, by a symme rical proof on $RBTs$, we show ha necessarily ei her he couple $((g_n-1),(g_n-1))$, $((g_n-1),(g_n-2))$ or $((g_n-2),(g_n-1))$ valua es he edges $(AA_1$, $BB_1)$. In any case, he MBT roo ed in A aking in o accoun hese cons rain s holds a mos $|V(MBT)| = 3 \times 2^{g_n-3}$ ver ices, which is always s ric ly less hen n (Figures (a), (b) and (c) in Figure 38).

Moreover, if $g_n \notin AB$, hen we know ha $g_n \in AA_1$ and $g_n \in BB_1$, and he MBT roo ed in A holds $|V(MBT)| = 3 \times 2^{g_n-3}$ ver ices, which is s ric ly less han n oo (cf. Figure (d) of Figure 38). So ha if $1 \notin AB$, hen $2 \in AA_1$ and $2 \in BB_1$.



(a) $2 \in AA_1$ and $3 \in BB_1$

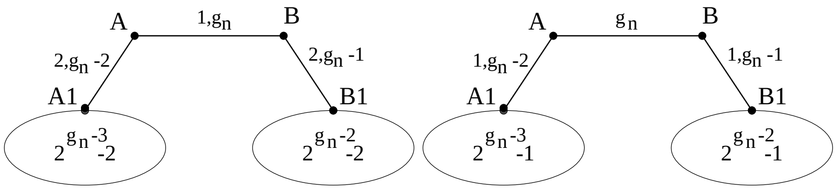
(b) $2 \in AA_1$ and $3 \in BB_1$

(c) $2 \in AA_1$ and $3 \in BB_1$

(d) $g_n \notin AB$

Figure 38: $MBTs$ roo ed in A

Now suppose $g_n \in AB$, and le 's consider he valua ion 1. If $1 \in AB$, we have proved ha necessarily $2 \in AA_1$ and $2 \in BB_1$. W.l.o.g., suppose ha $(g_n - 1) \notin AA_1$. Then he RBT roo ed in A holds $|V(RBT)| = 3 \times 2^{g_n - 3} - 2$ ver ices for any $g_n \geq 5$, and $|V(RBT)| < n$ in any case (cf. Figure (a) of Figure 39). If $1 \notin AB$, hen $1 \in AA_1$ and $1 \in BB_1$ and we ge a RBT roo ed in A wi h $|V(RBT)| = 3 \times 2^{g_n - 3}$ ver ices, which is also s ric ly less han n (cf. Figure (b) of Figure 39). The proof is hen comple e for $3 \times 2^{g_n - 3} + 1 \leq n \leq 2^{g_n - 1} - 1$.



(a) $1 \in AB$ and $(g_n - 1) \notin AA_1$ (b) $1 \notin AB$ and $(g_n - 1) \notin AA_1$

Figure 39: $RBTs$ roo ed in A

Finally, in he case $n = 2^p - 1$, le 's suppose we have $\{1, g_n\} \subseteq AB$. Then he firs par of he proper y yields $\{2, g_n - 1\} \subseteq AA_1$ and $\{2, g_n - 1\} \subseteq BB_1$. Then he MBT roo ed in A, as shown in Figure 40 holds $|V(MBT)| = 2^{g_n - 1} - 2$ ver ices, while $n = 2^{g_n - 1} - 1$. Hence $|V(MBT)| < n$, which shows 1 and g_n canno valua e AB a he same ime.

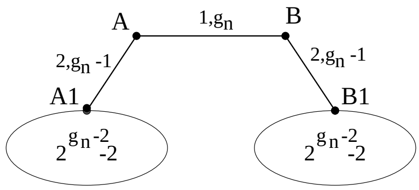


Figure 40: MBT roo ed in A where $\{1, g_n\} \subseteq AB$

Property 7 *For n odd such that $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, if there is a $(1, q)$-type edge AB in a MGG_n such that B is of degree q adjacent to C of degree 2 and C is adjacent to D, then :*

- $\{1, g_n\} \subseteq CD$ *;*
- *D cannot be of degree 2.*

Proof : The firs par of he lemma comes from Theorem 1, which yields ha if an edge AB is of ype $(1, q)$, hen $\{1, g_n\} \subseteq AB$. Consequen ly, as we consider a 1-por model, 1 and g_n canno valua e BC. Ye , C is of degree 2 and Theorem 2 says ha $\{1, g_n\} \subseteq BC \cup CD$, hence he resul .

Now le 's show ha D canno be of degree 2. If we suppose $deg(D) = 2$, as $\{1, g_n\} \subseteq CD$, Proper y 6 yields ha $\{2, g_n - 1\} \subseteq BC$ and $\{2, g_n - 1\} \subseteq DE$, where E is he ver ex adjacen o D which is no C, as shown in Figure 41. Hence, one can no ice ha $q \leq g_n - 2$. No e ha in he case $g_n = 5$, his could no happen since $q \geq 4$ (cf. Theorem 4) and $q \leq g_n - 2$, as s a ed above.

If we now look a he MBT roo ed in A, we show ha i holds $|V(MBT)| = 5\times 2^{g_n-4} - 2^{g_n-q} + 2$ ver ices, as shown in Figure 41. As $q \leq g_n - 2$, $|V(MBT)| \leq 5 \times 2^{g_n-4} - 2$, which is s ric ly less han n. Hence, D canno be of degree 2.

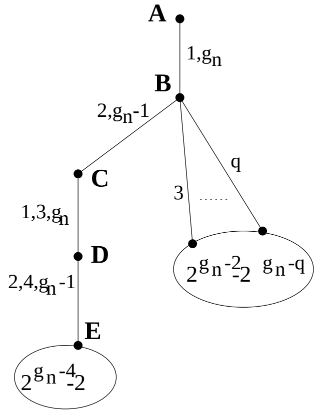


Figure 41: MBT roo ed in A

Lemma 4 *For all n odd such that $3 \times 2^{p-2} + 1 \leq n \leq 2^p - 1$ with $p \geq 4$, if there are s vertices of degree at least 3 in a MGG_n (vertices incident to $(1,q)$-type edges excluded), then there are at most $2s$ vertices of degree 2.*

Proof : In order o prove his lemma, we need o recall cer ain proper ies concerning MGG_n for such values of n. Those proper ies are he following :

- If here is a ver ex of degree 2 in a MGG_n hen 1 and g_n valua e i s inciden edges (cf. Theorem 2) ;
- A ver ex of degree 2 is adjacen o a mos 1 ver ex of degree 2 (cf. Theorem 6) ;
- If wo ver ices of degree 2, A and B, are adjacen , le 's call A_1 (resp.B_1) he ver ex adjacen o A (resp.B) which is no B (resp.A). Then we ge , hanks o Proper y 6 :
  1. If $1 \in AB$, hen $2 \in AA_1$ and $2 \in BB_1$ (we recall ha if $1 \notin AB$, hen Theorem 2 yields $1 \in AA_1$ and $1 \in BB_1$) (cons rain **C1**) ;
  2. If $g_n \in AB$, hen $(g_n - 1) \in AA_1$ and $(g_n - 1) \in BB_1$ (we recall ha if $g_n \notin AB$, hen Theorem 2 yields $g_n \in AA_1$ and $g_n \in BB_1$).
- There canno be any (1,3)- ype edge (cf. Theorem 4) ;
- If here is a $(1,q)$- ype edge AB, wi h $q \geq 4$, such ha A is of degree 1, hen if B is adjacen o C of degree 2, and C is adjacen o D, hen we know, hanks o Proper y 7 :
  1. $\{1, g_n\} \subseteq CD$;
  2. D canno be of degree 2 (cons rain **C2**).

We deno e by V_2 he se of ver ices of degree 2, and V_{3+} he se of ver ices of degree a leas 3. V_2 can be decomposed in hree par s :

- $V_{2,a}$ is he se of ver ices of degree 2 adjacen o ver ices of degree a leas 3;
- $V_{2,b}$ is he se of ver ices of degree 2 such ha any ver ex A of his se has a neighbour B of degree 2 and $1 \notin AB$;

- $V_{2,c}$ is he se of ver ices of degree 2 such ha any ver ex A of his se has a neighbour B of degree 2 and $1 \in AB$;

Clearly, $V_2 = V_{2,a} \oplus V_{2,b} \oplus V_{2,c}$. Analogously we can decompose V_{3+} in o wo dis inc subse s, which are $V_{1,q}$ and $\overline{V_{1,q}}$, where $V_{1,q}$ is he se of ver ices of degree a leas 3 having a neighbour of degree 1, and $\overline{V_{1,q}}$ is i s complemen in V_{3+}.

The proper ies s a ed above ensure us ha Φ and Ψ, defined below, are func ions, ha is any ver ex A has an image. Φ and Ψ are as follows :

$$\begin{aligned} \Phi \, : \, & V_{2,a} \cup V_{2,b} \longrightarrow \overline{V_{1,q}} \\ & A \longmapsto B \text{ such ha } 1 \in AB. \end{aligned}$$

Clearly, B is in $\overline{V_{1,q}}$ because a ver ex in $V_{1,q}$ already has he valua ion 1 on he $(1,q)$- ype edge. Moreover, Φ is injec ive because of he 1-por hypo hesis.

$$\begin{aligned} \Psi \, : \, & V_{2,c} \longrightarrow \overline{V_{1,q}} \\ & A \longmapsto B \text{ such ha } 1 \notin AB. \end{aligned}$$

No e ha B canno be in $V_{1,q}$, following cons rain **C2** ; which implies $B \in \overline{V_{1,q}}$. Moreover, Ψ is injec ive because if $1 \notin AB$, hen cons rain **C1** yields ha $2 \in AB$, and because we consider a 1-por model. As Ψ and Φ are injec ive, we ge he following inequali ies :

1. $|V_{2,a}| + |V_{2,b}| \leq |\overline{V_{1,q}}|$;
2. $|V_{2,c}| \leq |\overline{V_{1,q}}|$.

As he subse s of V_2 are dis inc and heir union is V_2 i self, adding hose inequali ies we ge $|V_2| \leq 2 \times |\overline{V_{1,q}}|$. Hence he resul .

Now, le 's prove he main heorem, ha is $G(n) \geq \lceil \frac{7n}{6} \rceil$. Le 's suppose we have r $(1,q)$- ype edges, where q may be variable. Le 's suppose we have s ver ices of degree a leas 3, such ha hose ver ices haven' go an inciden $(1,q)$- ype edge. Then here are $(n-s-2r)$ ver ices of degree 2. Analogously o he proof of Theorem 16, we ge wo inequali ies :

1. $n - s - 2r \leq 2s$;
2. $2 \times G(n) \geq 3s + 5r + 2 \times (n - s - 2r)$.

Those inequali ies lead o he following one : $G(n) \geq \frac{7n+r}{6}$. A worse, $r = 0$, hence he resul .

9.13 Proof of Theorem 21, page 11

In his sec ion, we are going o prove Proper y 8 and Lemmas 5, 6 and 7, which will help us o prove he heorem.

Property 8 *For $n = 2^p - 1$ with $p \geq 3$, if A is a vertex of degree at least 3, adjacent to B and C both of degree 2 such that B (resp. C) is adjacent to A and B_1 (resp. A and C_1), then we cannot have $deg(B_1) = deg(C_1) = 2$.*

Proof : Le 's suppose $deg(B_1) = deg(C_1) = 2$, as shown in Figure 42. As $n = 2^p - 1$, following Proper y 6 we know ha 1 and g_n do no valua e BB_1 (resp. CC_1) a he same ime. Bu if nei her 1 nor g_n valua e, say, BB_1, hen hey bo h mus valua e AB (cf. Theorem 2). As we considered a 1-por model, in ha case 1 and g_n don' valua e AC. As a consequence, following Theorem 2, 1 and g_n mus valua e CC_1, hence here is a con radic ion. Then he only possible cases are when $1 \in BB_1$ and $g_n \in CC_1$, or he o her way round. Suppose, w.l.o.g., ha $1 \in BB_1$ and $g_n \in CC_1$. Hence, Proper y 6 implies $\{2, g_n\} \subseteq AB$, $\{2, g_n\} \subseteq B_1B_2$, $\{1, g_n - 1\} \subseteq AC$ and $\{1, g_n - 1\} \subseteq C_1C_2$ (we refer o Figure 42 for a be er unders anding of hese s a emen s). Then he MBT roo ed in A holds $|V(MBT)| = 7 \times 2^{g_n - 4} - 2$ ver ices, which is s ric ly less han $n = 2^{g_n - 1} - 1$ for any $g_n \geq 4$. As a consequence we canno have bo h B_1 and C_1 of degree 2.

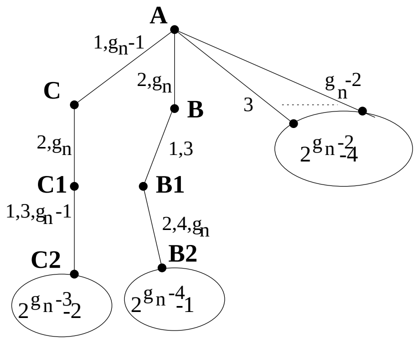


Figure 42: MBT roo ed in A

Lemma 5 *For all $n = 2^p - 1$ with $p \geq 4$, the only $(1, q)$-type edge that can exist in a MGG_n is for $q = p$.*

Proof : Theorem 1 yields ha if AB is a $(1, q)$- ype edge, hen $AB = \{1, g_n\}$. Therefore, as $g_n = p + 1$ and AB has 2 valua ions, we have $q \leq p$ (we recall ha he edge AB is aken in o accoun in he value of q).

Suppose we have a $(1, q)$- ype edge AB, and le 's call he ver ices adjacen o B (A excluded) $B_2 \ldots B_q$. Now le 's build he MBT roo ed in A, as shown in Figure 43. Clearly, i holds $|V(MBT)| = 2^{g_n - 1} - 2^{g_n - q} + 2$ ver ices. As $q \neq g_n$ since $g_n \in AB$, he only case for which $|V(MBT)| \geq n$ is when $q = g_n - 1$ (and, in ha case, $|V(MBT)| = 2^{g_n - 1} = 2^p$). Since $g_n = p + 1$, we ge he resul .

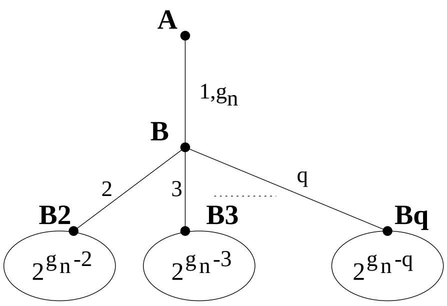


Figure 43: MBT roo ed in A where AB is a $(1, q)$- ype edge

Lemma 6 *For all $n = 2^p - 1$ with $p \geq 4$, if we have a $(1, p)$-type edge AB in a MGG_n such that A is of degree 1, then B can't be adjacent to any vertex of degree 2.*

Proof : Le 's consider a $(1, p)$- ype edge, and suppose ha B_j is of degree 2. Le 's call C_j he ver ex adjacen o B_j which is no B, as shown in Figure 44. Then Theorem 2 yields ha 1 and g_n valua e $B_j C_j$. If we hen consider he MBT roo ed in A, we ge $|V(MBT)| = 2^{g_n - 1} - 2^{g_n - j - 1} + 1$, which is s ric ly less han n for any $j \neq g_n - 1$.

In he case $j = g_n - 1$, however, considering he RBT roo ed in A as shown in Figure 45, we ge $|V(RBT)| = 3 \times 2^{g_n - 3}$, which is s ric ly less han n for any $g_n \geq 4$.

In all cases, we show ha here is a leas a RBT or a MBT ha canno hold n or more ver ices. Then here is no ver ex of degree 2 adjacen o B in a MGG_n such ha $n = 2^p - 1$.

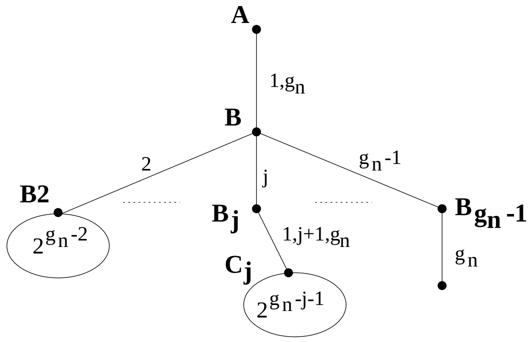


Figure 44: MBT roo ed in A, where B_j is of degree 2

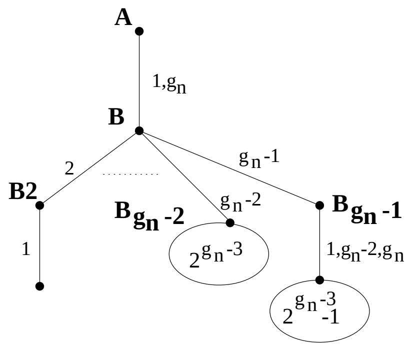


Figure 45: RBT roo ed in A where B_{g_n-1} is of degree 2

Lemma 7 *For all $n = 2^p - 1$ with $p \geq 4$, if there are s vertices of degree at least 3 in a MGG_n (vertices incident to $(1,p)$-type edges excluded), then there are at most $\frac{3s}{2}$ vertices of degree 2.*

Proof : We need firs o recall differen proper ies concerning MGG_n for such a value of n. They are he following :

- If here are $(1,q)$- ype edges, hen necessarily $q = p$ (cf. Lemma 5) ;
- If here is a $(1,p)$- ype edge AB where B is of degree p, hen B canno be adjacen   o any ver ex of degree 2 (cf. Lemma 6) ;
- If here is a ver ex A of degree 2, adjacen   o B and C, hen $\{1, g_n\} \subseteq AB \cup AC$ (cf. Theorem 2) ;
- If A is of degree 2, adjacen   o B and C, hen we canno have $deg(B) = deg(C) = 2$ (cf. Theorem 6) ;
- If wo ver ices of degree 2, A and B, are adjacen , le 's call A_1 (resp.B_1) he ver ex adjacen   o A (resp.B) which is no B (resp.A). Then we ge , hanks o Proper y 6 :
  1. If $1 \in AB$, hen $2 \in AA_1$ and $2 \in BB_1$ (we recall ha if $1 \notin AB$, hen Theorem 2 implies $1 \in AA_1$ and $1 \in BB_1$) ;
  2. If $g_n \in AB$, hen $(g_n - 1) \in AA_1$ and $(g_n - 1) \in BB_1$ (we recall ha if $g_n \notin AB$, hen Theorem 2 implies $g_n \in AA_1$ and $g_n \in BB_1$) ;

3. We canno have $1 \in AB$ and $g_n \in AB$.

- If A is a ver ex of degree a leas 3 adjacen o wo ver ices of degree 2, B and C, such ha B (resp. C) is adjacen o A and B_1 (resp. A and C_1), hen we canno have $deg(B_1) = deg(C_1) = 2$ (cons rain **C1**, cf. Proper y 8).

Le V_2 deno e he se of ver ices of degree 2 and V_{3+} he se of ver ices of degree a leas 3. Le us decompose V_2 and V_{3+} as i has been done in Lemma 4, and le us define he following func ions :

$\Phi : V_{2,c} \longrightarrow W_1$
$A \longmapsto B$ such ha A is adjacen o a ver ex $A' \neq B$ of degree 2 and $g_n \in AB$.

$\Psi : V_{2,b} \longrightarrow W_2$
$A \longmapsto B$ such ha A is adjacen o a ver ex $A' \neq B$ of degree 2 and $1 \in AB$.

Clearly, W_1 and W_2 are subse s of V_{3+}. Moreover, $W_1 \cap W_2 = \emptyset$, hanks o cons rain **C1**.
Le us hen define wo more func ions :

$\Sigma : V_{2,a} \longrightarrow \overline{V_{1,q}} - W_2$
$A \longmapsto B$ such ha $1 \in AB$.

$\Sigma' : V_{2,a} \longrightarrow \overline{V_{1,q}} - W_1$
$A \longmapsto B$ such ha $g_n \in AB$.

Clearly, any $B = \Sigma(A)$ is in $\overline{V_{1,q}} - W_2$, because o herwise wo edges inciden o a ver ex of degree a leas 3 would be valua ed by 1. Analogously, any $B = \Sigma'(A)$ is in $\overline{V_{1,q}} - W_1$.

Firs , no e ha any ver ex A has an image by any of he func ions above, provided i is in he righ subse . Moreover, Φ, Ψ, Σ and Σ' are all injec ive, hanks o he 1-por model. We hen ge wo inequali ies, depending whe her we look a Σ or Σ' :

1. $|V_2| \leq |\overline{V_{1,q}}| + |W_1|$;
2. $|V_2| \leq |\overline{V_{1,q}}| + |W_2|$.

If we add hem, we ge $2 \times |V_2| \leq 2 \times |\overline{V_{1,q}}| + |W_1| + |W_2|$. As $|W_1| + |W_2| \leq |\overline{V_{1,q}}|$, we ge $2 \times |V_2| \leq 3 \times |\overline{V_{1,q}}|$, hence he resul .

Finally, le us come back o he main resul , ha is $G(n) \leq \lceil \frac{6n}{5} \rceil$. Le 's suppose ha we have r $(1,p)$- ype edges, and s ver ices of degree a leas 3 (ver ices inciden o $(1,p)$- ype edges excluded). Consequen ly, we have $(n - s - 2r)$ ver ices of degree 2. Hence he firs inequali y : $n - s - 2r \leq \frac{3s}{2}$. Moreover, he sum of he ver ices degrees is wice he number of edges, ha is $2 \times G(n) \geq 3s + (p+1) \times r + 2 \times (n - s - 2r)$. Finally, we ge he following inequali y : $G(n) \geq \frac{6n}{5} + \frac{5(p+1)-24}{10} \times r$. We recall ha $g_n = p+1$ in ha case. A worse, here is no $(1,p)$- ype edge and we ge he resul .

9.14 Proof of Theorem 23, page 11

We already know ha $G(15) \geq 18$ hanks o Theorem 21, and ha $G(15) \leq 19$ hanks o Corollary 3. Le us recall o hose wo proper ies, coming from Lemma 7 :

1. If here are q ver ices of degree a leas 3, ver ices inciden o $(1,4)$- ype edges excluded, hen here can be a mos $\frac{3q}{2}$ ver ices of degree 2 ;
2. $G(n) \geq 18 + \frac{r}{10}$, where r is he number of $(1,4)$- ype edges.

In ha case, suppose $G(15) = 18$. Then here is no (1,4)- ype edge. As he only ver ices of degree 1 ha can exis are ver ices inciden o (1,4)- ype edges, i means ha we canno have any ver ex of degree 1. Hence, if we have s ver ices of degree a leas 3, hen we have exac ly $(n - s)$ ver ices of degree 2. Moreover, we know ha $n - s \leq \frac{3s}{2}$ (Lemma 7), ha is $s \geq 6$. As we supposed $G(15) = 18$, he only possible case is when we have 6 ver ices of degree 3 and 9 ver ices of degree 2.

Le 's now consider ha case, and recall he proof of Lemma 7, where we had hose wo inequali ies :

- $|V_2| \leq |\overline{V_{1,q}}| + |W_1|$;
- $|V_2| \leq |\overline{V_{1,q}}| + |W_2|$.

Those inequali ies imply $|W_i| \geq 3$ wi h $i \in \{1,2\}$. As hose wo subse s are dis inc and $|\overline{V_{1,q}}| = 6$, we ge $|W_1| = |W_2| = 3$. However $|V_{2,b}|$ and $|V_{2,c}|$ are even, by defini ion. We knew, hanks o he func ion Ψ, ha $|V_{2,b}| \leq |W_2|$. We can hen say ha $|V_{2,b}| \leq |W_2| - 1$. Moreover, $|V_2| = |\overline{V_{2,a}}| + |V_{2,b}| + |V_{2,c}|$. Hence we have, hanks o Ψ, Φ and Σ, he following inequali y : $|V_2| \leq |\overline{V_{1,q}}| + |W_1| - 1$, ha is $9 \leq 8$. Then his case canno lead o a MGG_{15} ; herefore $G(15) \neq 18$ and $G(15) = 19$.