Minimum Gossip Digraphs

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Abstract

Gossiping is a problem of information dissemination described in a group of individuals connected by a communication network, where every individual in the network knows a unique item of information and needs to communicate it to everyone else. This communication pattern finds its main applications in the field of interconnection networks for parallel and distributed architecture. Previous papers have investigated ways to construct sparse graphs (networks) in which this process can be completed in minimum time. In this paper, we consider the gossiping problem in directed graphs. We discuss about the minimum time necessary to achieve gossiping in digraphs, and describe several techniques to construct sparse digraphs on n vertices in which gossiping can be completed in minimum time. For some small values of n ($1 \leq n \leq 8$), these techniques produce the sparsest possible digraphs of this type (called minimum gossip digraphs). For other values of n, these techniques produce the sparsest known digraphs of this type.

1 Introduction

Gossiping refers to the process of dissemination of information in a communication network where each node holds a piece of information and needs to transmit it to every other node in the network. This is achieved by placing calls over the communication lines of the network. We consider a *constant-time*, *1-port*, *half-duplex* model, that is each call requires one unit of time and at any given time, a vertex can communicate with at most one of its neighbours. Moreover, if there is a communication between vertices u and v, then either u sends its information to v or v sends its information to u. Hence, we will work on directed and strongly connected graphs, without loops or multiple edges. Given any of those digraphs, G, $\vec{g}(G)$ denotes the minimum amount of time necessary to gossip in G, or the *gossip time* of G. Let $\vec{g}_n = \vec{g}(K_n^*)$ for any n, where K_n^* is the complete symmetric digraph of order n. A *gossip digraph* is a digraph able to achieve gossiping in $\vec{g}_n$ rounds. However, it is not necessary to consider K_n^* to get a gossip digraph. We then call *Minimum Gossip Digraph* of order n, or MGD_n, any gossip digraph with a minimum number of edges. This number is denoted by $\vec{G}(n)$. From an application perspective, MGDs represent the cheapest possible communication networks (i.e. with a minimum number of communication lines) in which gossiping can be achieved in minimum time.

This paper presents new results concerning both the time to gossip in the complete digraph and bounds on the number of directed edges of MGDs.

Section 2 provides a quick survey of gossiping in directed graphs, which focuses on the minimum time to gossip in digraphs ; this is followed by some improvements about the knowledge of the exact minimum time necessary to achieve gossiping. Sections 3 and 4 give respectively lower and upper bounds on the size of minimum gossip digraphs, while Section 5 provides a collection of results for small values of n.

2 Gossip Time in the Half-Duplex Model

In [EM89], Even and Monien studied the minimum amount of time necessary to achieve gossiping in a directed graph. Their result is the following :

Theorem 1 ([EM89]) *For any n, let k be the least integer such that $F(k) \geq \lfloor \frac{n}{2} \rfloor$, where $F(p)$ is the p-th Fibonacci number (we recall that $F(1) = F(2) = 1$ and $F(p+2) = F(p+1) + F(p)$ for all $p \geq 1$). The minimum time necessary to gossip in a n-vertices digraph is :*

- *$k+1$ if $n = 2 \cdot F(k)$;*
- *k or $k+1$ if n is even and $n \neq 2 \cdot F(k)$;*
- *k, $k+1$ or $k+2$ if n is odd and $n \geq 7$.*

Moreover, they proved the following Theorem thanks to a nice proof using matrix norms.

Theorem 2 ([EM89]) *For all n, $\vec{g}_n \geq 2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil$ where $\rho = \frac{1+\sqrt{5}}{2}$.*

From this, we can derive the following Observation.

Observation 1 *For any $n \geq 5$, $\vec{g}_n \leq n-1$.*

Proof : Let k be the least integer such that $F(k) \geq \lfloor \frac{n}{2} \rfloor$. By Theorem 1, $\vec{g}_n \leq k+2$. Hence we have $F(\vec{g}_n - 3) \leq F(k-1) < \lfloor \frac{n}{2} \rfloor \leq \frac{n}{2}$. It is easy to see that for any $m \geq 8$, $F(m-3) \geq \frac{m}{2}$. Hence, for any $\vec{g}_n \geq 8$, we have $\frac{\vec{g}_n}{2} \leq F(\vec{g}_n - 3) < \frac{n}{2}$, i.e. $\vec{g}_n \leq n-1$. Theorem 2 yields $\vec{g}_n \geq 2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil$ where $\rho = \frac{1+\sqrt{5}}{2}$. Hence for all n such that $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil \geq 8$ we will have $\vec{g}_n \geq 8$, and consequently $\vec{g}_n \leq n-1$. Standard calculations show that for any $n \geq 23$, we have $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil \geq 8$, which implies $\vec{g}_n \leq n-1$.

Moreover, a case by case analysis shows that for any $6 \leq n \leq 22$, $\vec{g}_n \leq n-1$ too (the complete proof is omitted here ; however, we refer to Table 1 for results concerning $n \leq 16$). □

However, we will see that it is possible to get more information on the gossip time in the general case. First, let us show the following Property, due to Liestman, Raspaud and Shermer [LRS97].

Property 1 *For all $m \geq 0$, $\vec{g}_{2m+2} \leq \vec{g}_{2m+1}$. Moreover, if $\vec{g}_{2m+1} = k$ where k is the least integer such that $F(k) \geq \lfloor \frac{2m+1}{2} \rfloor$, then $\vec{g}_{2m+2} = k$.*

Proof : Suppose we have a digraph with $2m+1$ vertices that can achieve gossiping within k rounds. Then, as $2m+1$ is odd, we know that during the first round (resp. the last round), there is one vertex v (resp. w) that does not communicate. Hence, we can add a vertex u and directed edges (u, v) and (w, u) such that the new digraph (with $2m+2$ vertices) can gossip within k rounds too. It suffices to communicate along (u, v) at round 1 and along (w, u) at round k. Hence $\vec{g}_{2m+2} \leq k$, that is $\vec{g}_{2m+2} \leq \vec{g}_{2m+1}$, since we supposed $\vec{g}_{2m+1} = k$.

Now suppose $\vec{g}_{2m+1} = k$, where k is the least integer such that $F(k) \geq \lfloor \frac{2m+1}{2} \rfloor$. Then, by definition of k, we have $F(k-1) < m$. Now, by Theorem 1, we see that $\vec{g}_{2m+2} \geq k'$, where k' is the least integer such that $F(k') \geq m+1$. Hence $F(\vec{g}_{2m+2}) \geq m+1$, and we have $F(\vec{g}_{2m+2}) \geq m+1 > m > F(k-1)$, that is $\vec{g}_{2m+2} \geq k$. Since we knew $\vec{g}(2m+2) \leq k$, we get the equality. □

Thanks to Even and Monien, we can derive a certain number of values of n for which we have $\vec{g}_n \neq k$. This is the purpose of the following Theorem.

Theorem 3 *For all n, let k be the smallest integer such that $F(k) \geq \lfloor \frac{n}{2} \rfloor$. Let $\rho = \frac{1+\sqrt{5}}{2}$ and $\overline{\rho} = 1 - \rho = \frac{1-\sqrt{5}}{2}$. Then :*

(a) If n and k are even and $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1)$, then $\vec{g}_n = k+1$;

(b) If n is even, k is odd and $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) + \frac{2}{2\rho+1} \cdot (-\overline{\rho})^k$, then $\vec{g}_n = k+1$;

(c) If n is odd, k is even and $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) - 1$, then $\vec{g}_n = k+1$ or $\vec{g}_n = k+2$;

(d) If n and k are odd and $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) + \frac{2}{2\rho+1} \cdot (-\overline{\rho})^k - 1$, then $\vec{g}_n = k+1$ or $\vec{g}_n = k+2$.

Proof : The aim is to show that, in each case, for such values of n, $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, and the result follows directly by Theorems 1 and 2. For this, we need to recall the following equalities and inequalities : $F(k) = \frac{\rho^k - (\overline{\rho})^k}{\sqrt{5}}$, $2\rho + 1 = \rho^3$ and $-1 < \overline{\rho} < 0$.

The proof, in each of the four cases, relies on the same arguments. Let us show, for example, the result in the case n even and k even. In that case, $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1)$ becomes $\frac{n}{2} \geq \frac{\rho^{k+1} - (\overline{\rho})^{k+1}}{\rho^3}$ by the first two equalities. Since k is even, we have $(\overline{\rho})^{k+1} < 0$, that is $\frac{n}{2} > \rho^{k-2}$. Hence $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, which, by Theorems 1 and 2, yields $\vec{g}_n = k+1$. The same method applies for the three other cases (the proof is omitted here) ; we can show, by standard calculations, that we have, in every case, $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, which implies $\vec{g}_n \geq k+1$ by Theorems 1 and 2. □

There are values of (odd) n for which we can say that $\vec{g}_n \neq k+2$. Let us first give the following property, due to Liestman, Raspaud and Shermer [LRS97], which will be useful to get those values.

Property 2 *For all integers n and $p \geq n$, let i be the smallest integer such that $p \leq 2^i n$ (that is, $i = \lceil \log_2(\frac{p}{n}) \rceil$). In that case, we have $\vec{g}_p \leq \vec{g}_n + 2i$.*

Proof : Let us first consider the case $p \leq 2n$. First, if $p = 2n$ (case **C1**), is not difficult to see that we have $\vec{g}_{2n} \leq \vec{g}_n + 2$ for any n. For this, take a gossip digraph G with n vertices and, to each vertex u of G we add a vertex u' and two directed edges (u', u) and (u, u'). If we communicate along (u', u) and (u, u') respectively at rounds 1 and $\vec{g}_n + 2$, while gossiping in G takes place between rounds 2 and $\vec{g}_n + 1$, we get a digraph of order $2n$ able to achieve gossiping in $\vec{g}_n + 2$ rounds. Now, if $p < 2n$ (case **C2**), the result remains true : it suffices to add only $(p - n)$ vertices (and their corresponding directed edges) instead of n.

If $2^{i-1}n < p \leq 2^i n$ with $i \geq 2$, then apply the method of case **C1** recursively ($(i-1)$ times), using for each step of the recursion the new digraph obtained. Then we get a digraph with $2^{i-1}n$ vertices able to gossip in $\vec{g}_n + 2(i-1)$ rounds. Now add $(p - 2^{i-1}n)$ vertices, using the method of case **C2**. We then obtain a gossip digraph with p vertices, able to gossip in $\vec{g}_n + 2i$ rounds. □

Thanks to Property 2, it is not difficult to get the following Proposition.

Proposition 1 *Let n be an odd number, and k the least integer such that $F(k) \geq \lfloor \frac{n}{2} \rfloor$. If $k \geq 7$ and $n \leq 4 \cdot F(k-2) - 1$, then $k \leq \vec{g}_n \leq k+1$.*

Proof : Note first that, by definition of k, we have $F(k-1) < \lfloor \frac{n}{2} \rfloor$, that is $n > 2 \cdot F(k-1) + 1$. Proposition 1 holds when $n \leq 4 \cdot F(k-2) - 1$, hence it is legitimate to wonder whether $2 \cdot F(k-1) + 1 < 4 \cdot F(k-2) - 1$. Since $F(p) = F(p-1) + F(p-2)$ for any $p \geq 3$, we see that the above inequality resolves to $F(k-4) > 1$, hence it holds for any $k \geq 7$.
By Theorem 1, we know that $k \leq \vec{g}_n \leq k+2$. Our aim here is to show that $\vec{g}_n \leq k+1$. For this, let $n' = 2 \cdot F(k-2)$. By Theorem 1, we know that $\vec{g}_{n'} = k-1$. Note that $n \geq n'$, since $n > 2 \cdot F(k-1) + 1$ by definition and $n' = 2 \cdot F(k-2)$. By Property 2 (with $i = 1$), we know that for any $n' \leq n \leq 2n'$, $\vec{g}_n \leq k+1$. That is, for any $n \leq 4 \cdot F(k-2)$, $\vec{g}_n \leq k+1$. □

3 Lower bounds for $\vec{G}(n)$

In order to prove Proposition 2, let us first prove the following Observation.

Observation 2 *The number of rounds r_n necessary to gossip in a n-circuit is at least n.*

Proof : Let G be a n-circuit. Then there are two vertices, u and v, such that the directed distance $\vec{d}(u, v) = n - 1$. Hence $r_n \geq n - 1$. Now suppose that $r_n = n - 1$, and let u' be the successor of u. In that case, there is a communication along the directed edge (u, u') at round 1, in order for u to inform v. But that implies that u' cannot start sending its information before round 2 (due to the 1-port model), and therefore u' will be unable to reach u. Hence $r_n \geq n$. □

Proposition 2 *For all $n \geq 6$, $\vec{G}(n) \geq n + 1$.*

Proof : First, we know that in a MGD_n each vertex must have indegree and outdegree at least equal to 1, otherwise gossiping cannot be achieved. Hence $\vec{G}(n) \geq n$. Suppose now that $\vec{G}(n) = n$. In that case, the corresponding digraph is necessarily a n-circuit. Observation 2 then yields that $\vec{g}_n \geq n$. However, by Observation 1 we have $\vec{g}_n \leq n - 1$ for any $n \geq 5$, a contradiction. Hence, for any $n \geq 6$, a n-circuit cannot be a gossip digraph, and we have $\vec{G}(n) \geq n + 1$. □

Theorem 4 *For any n, let i_m be the greatest integer i such that $i + 2^{\vec{g}_n - i} \geq n + 1$, with $i \leq \vec{g}_n - 1$. Then we have $\vec{G}(n) \geq n + \lceil \frac{n}{3i_m + 2} \rceil$.*

Proof : First, note that in the following we will denote the indegree (resp. the outdegree) of a vertex u by $d^-(u)$ (resp. $d^+(u)$). The *degree* of u, $\deg(u)$, is defined by : $\deg(u) = d^+(u) + d^-(u)$.

Now, let us count the maximum number i_m of adjacent vertices of degree 2 in a MGD_n. We recall that in a MGD_n, every vertex u of degree 2 is such $d^+(u) = d^-(u) = 1$. Suppose we have a directed path in a MGD_n with i vertices of degree 2, as shown in Figure 1. Note first that necessarily $i \leq \vec{g}_n - 1$, otherwise u_2 could not inform u_1. It is not difficult to see that the "first" vertex of degree 2, u_1, can broadcast its information to up to $i + 2^{\vec{g}_n - i}$ vertices. But, in that case, u_2 can only broadcast its information to up to $i + 2^{\vec{g}_n - i} - 1$ vertices. Hence i_m is not bigger than the greatest integer i such that $i + 2^{\vec{g}_n - i} \geq n + 1$ (with $0 \leq i \leq \vec{g}_n - 1$). Note that i_m is always determined since $f(i) = i + 2^{\vec{g}_n - i} - (n+1)$ is decreasing for any $0 \leq i \leq \vec{g}_n - 1$.

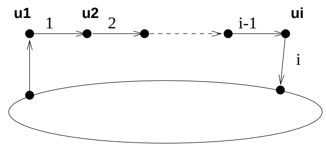

Figure 1: Adjacency of vertices of degree 2

Now let us fix n, the number of vertices, and let us suppose that $\vec{G}(n) = n + p$ with $p < \frac{n}{3i_m+2}$. In that case, any MGD_n holds at least one vertex of degree 2. Let V_3 be the set of vertices of degree at least 3 in a MGD_n, with $x = |V_3|$. Since $p \geq 1$, $x \geq 1$. Note also that $x \leq 2p$: indeed, if we sum the degrees over all the vertices of the digraph, we get $S = \sum_{u \in V(G)} d^+(u) + d^-(u) = 2(n+p)$. Moreover, as there are x vertices of degree at least 3, we also have $S \geq 2(n-x) + 3x$.

We have $\sum_{u \in V_3} d^+(u) + d^-(u) = 2(n+p) - 2(n-x) = 2p + 2x$ too. Hence P, the number of directed paths of vertices of degree 2 in the digraph, satisfies the inequality : $1 \leq P \leq p + x$. Since we have P paths of vertices of degree 2, we also have the following inequality : $i_m \cdot P \geq n - x$. Considering that $P \leq p + x$, we then have : $i_m \geq \frac{n-x}{p+x}$. Since in every case we have $1 \leq x \leq 2p$, i_m must verify $i_m \geq \frac{n-2p}{3p}$, that is $p \cdot (3i_m + 2) \geq n$. However, we supposed $p < \frac{n}{3i_m+2}$, a contradiction. Hence $\vec{G}(n) \geq n + \lceil \frac{n}{3i_m+2} \rceil$. □

Example 1 *For $n = 13$ and $n = 14$, we have $\vec{g}_n = 7$ by Theorem 12. In that case, $i_m = 3$, and $\vec{G}(13) \geq 13 + \lceil \frac{13}{11} \rceil$, that is $\vec{G}(13) \geq 15$. Similarly, $\vec{G}(14) \geq 16$. Note that Theorem 4 above has been used to determine many of the lower bounds for $\vec{G}(n)$ in Table 1 of Section 5.*

4 Upper bounds for $\vec{G}(n)$

There are different ways to get upper bounds for $\vec{G}(n)$. The first one is to find families of digraphs that achieve gossiping within $\vec{g}_n$ rounds, and have as few edges as possible. A second way is to build gossip digraphs from existing (minimum) gossip digraphs. We study both methods in the following. First, we prove Proposition 3, which gives a general upper bound for $\vec{G}(n)$.

Proposition 3 $\vec{G}(n) \leq \begin{cases} \frac{n \cdot \vec{g}_n}{2} & \text{for all even } n \text{ ;} \\ \frac{(n-1) \cdot \vec{g}_n}{2} & \text{for all odd } n. \end{cases}$

Proof : We know that, for every vertex u in a MGD_n, there must be a communication along each incident edge of u during at least one round (otherwise, this edge could be removed without affecting the gossip scheme). Hence we have $d^+(u) + d^-(u) \leq \vec{g}_n$. If we sum this inequality over all the vertices of the digraph, we get $\vec{G}(n) \leq \frac{n \cdot \vec{g}_n}{2}$. Moreover, in the odd case, we know that, at each round, there is at least one vertex which does not communicate its information to the others, hence we get $\sum_{u \in V(G)} d^+(u) + d^-(u) \leq n \cdot \vec{g}_n - \vec{g}_n$. □

Theorem 5 *For all even n such that $\vec{g}_n = k+1$, where k is the least integer such that* $F(k) \geq \frac{n}{2}$: $\vec{G}(n) \leq \begin{cases} \frac{n(k-1)}{2} & \text{if } n \leq 2F(k) - 2 \text{ ;} \\ \frac{nk}{2} & \text{if } n = 2F(k). \end{cases}$

Proof : Note first that since we supposed k being the least integer such that $F(k) \geq \frac{n}{2}$, we necessarily have $F(k-1) < \frac{n}{2}$, that is $n \geq 2 \cdot F(k-1) + 2$. Note also that in [EM89], Even and Monien gave an algorithm to prove that it is possible to achieve gossiping in $k+1$ rounds for any even n. We define here the family of *Fibonacci digraphs* underlying their construction.

Definition 1 (Fibonacci digraph) *The* Fibonacci digraph *on $n \geq 2$ vertices (n even) and of dimension $1 \leq \Delta \leq q$ (for any q such that $F(q) \leq \frac{n}{2}$) is denoted $F_{\Delta,n}$. The vertices of $F_{\Delta,n}$ are the couples (i,j) with $i = 1,2$ and $0 \leq j \leq \frac{n}{2} - 1$. For every j, $0 \leq j \leq \frac{n}{2} - 1$, there is a directed edge from vertex $(1,j)$ to vertex $(2, j + F(p) - 1 \mod \frac{n}{2})$ for any odd p and a directed edge from vertex $(2,j)$ to vertex $(1, j - F(p) + 1 \mod \frac{n}{2})$ for any even p, with $p = 1, 2, \ldots \Delta$.*

For $1 \leq p \leq \Delta$, a directed edge which connects a vertex $(1,j)$ to a vertex $(2, j + F(p) - 1 \mod \frac{n}{2})$ (resp. a vertex $(2,j)$ to a vertex $(1, j - F(p) + 1 \mod \frac{n}{2})$) is said to be *in dimension* p. We also note that in $F_{\Delta,n}$, the number of arcs is equal to $\frac{\Delta n}{2}$. For a better understanding of a Fibonacci digraph, we refer to Figure 2, where $F_{5,14}$ and $F_{5,10}$ are shown.

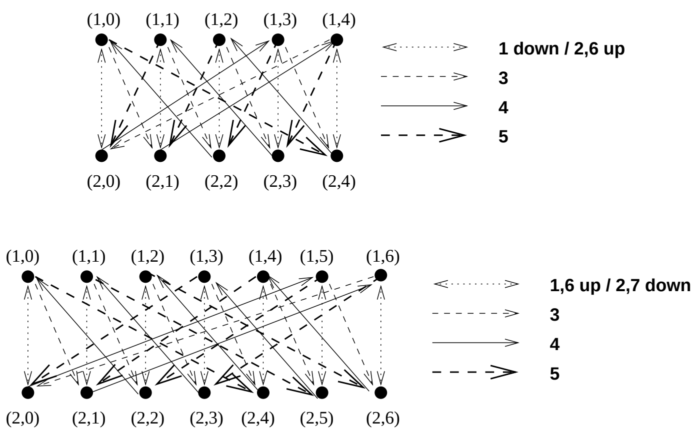

Figure 2: Fibonacci digraphs : (a) $F_{5,10}$ and a gossip scheme (b) $F_{5,14}$ and a gossip scheme

Even and Monien [EM89] showed that for any even n, it is possible to achieve gossiping in $k+1$ rounds, where k is the least integer such that $F(k) \geq \frac{n}{2}$. The digraph underlying their construction is in that case the Fibonacci digraph $F_{k+1,n}$, with $\frac{n(k+1)}{2}$ arcs. Here, we show that if $n = 2 \cdot F(k)$, then $F_{k,n}$ is a gossip digraph (that is, we save $\frac{n}{2}$ arcs), and that if $n \leq 2 \cdot F(k) - 2$ (with $\vec{g}_n = k+1$), then $F_{k-1,n}$ is a gossip digraph (that is, we save n arcs).

Let us show first the latter. For this, let the gossip scheme be the following : the vertices of $F_{k-1,n}$ communicate along dimension p for every $1 \leq p \leq k-1$. Then, indifferently, the vertices communicate again along dimensions 1 and 2 (or along dimensions 2 and 1) during rounds k and $(k+1)$ (cf. Figure 2(b)). Clearly, this gossip scheme respects the 1-port hypothesis, since the set of arcs used at any round $1 \leq p \leq k+1$ forms a perfect matching of $F_{k-1,n}$. Now let us see how a vertex u broadcasts its information in $F_{k-1,n}$ following this scheme ; suppose first that $u = (1,0)$.

Property 3 *For any $1 \leq m \leq \lfloor \frac{k-1}{2} \rfloor$, after $p = 2m$ rounds, vertex $(1,0)$ has broadcast its information to the $F(p)-1$ "last" vertices among the $(1,j)$ (that is, every vertex of the form $(1, \frac{n}{2} - j)$ with $1 \leq j \leq F(p)-1$) and to the $F(p-1)$ "first" vertices among the $(2,j)$ (that is, every vertex of the form $(2,j)$ with $0 \leq j \leq F(p-1)-1$).*

Proof : By induction on m. When $m = 1$, that is $p = 2$, we see that it is true. Now suppose it is true for m, and let us show it is true for $m+1$. During round $p+1 = 2m+1$, there is a communication along arcs on (odd) dimension $p+1$. That is, every informed vertex of the form $(1,j)$ will inform vertex $(2, j + F(p+1) - 1 \bmod \frac{n}{2})$. Hence any vertex of the form $(1, \frac{n}{2} - j)$ will inform vertex $(2, F(p+1) - 1 - j)$ with $1 \leq j \leq F(p) - 1$. Thus we can rewrite this as follows : every vertex of the form $(2,j)$ with $F(p-1) \leq j \leq F(p+1) - 2$ is newly informed. Moreover, vertex $(1,0)$ informs vertex $(2, F(p+1)-1)$ at round $p+1$. Hence, on the whole, vertices $(2,j)$, with $0 \leq j \leq F(p+1)-1$ are informed.

During round $p+2 = 2m+2$, there is a communication along arcs on (even) dimension $p+1$. That is, each vertex $(2,j)$ (with $0 \leq j \leq F(p+1)-1$) informs vertex $(1, j - F(p+2) + 1 \bmod \frac{n}{2})$ at round $p+2$. Thus every vertex of the form $(1, \frac{n}{2} - j)$ with $F(p) \leq j \leq F(p+2)-1$ is informed. On the whole, we see that all vertices of the form $(1, \frac{n}{2} - j)$, with $1 \leq j \leq F(p+2)-1$, are informed. Consequently, Property 3 is proved by induction on m. □

Property 4 *For any $1 \leq m \leq \lfloor \frac{k-2}{2} \rfloor$, after $p = 2m+1$ rounds, vertex $(1,0)$ has broadcast its information to the $F(p-1)-1$ "last" vertices among the $(1,j)$ and to the $F(p-1)$ "first" vertices among the $(2,j)$.*

Proof : Similar to Proof of Property 3. □

We see that after $k-1$ rounds, we have :

- If k is even : vertex $(1,0)$, vertices $(1, \frac{n}{2} - j)$ $(1 \leq j \leq F(k-2)-1)$, and vertices $(2,i)$ $(0 \leq i \leq F(k-1)-1)$ are informed ;
- If k is odd : vertex $(1,0)$, vertices $(1, \frac{n}{2} - j)$ $(1 \leq j \leq F(k-1)-1)$, and vertices $(2,i)$ $(0 \leq i \leq F(k-2)-1)$ are informed.

Since there is a communication along dimensions 1 and 2 at rounds k and $k+1$, we see that every vertex is informed iff $F(k-2) \geq \frac{n}{2} - F(k-1) + 1$, that is iff $n \leq 2F(k) - 2$, which is true by hypothesis. Hence $(1,0)$ can broadcast its information in $F_{k-1,n}$ in $k+1$ rounds, following the given scheme. It is easy to see that actually any vertex of the form $(1,j)$ is able to broadcast in $k+1$ rounds in $F_{k-1,n}$. This is also the case for any vertex of the form $(2,j)$, thanks to the communications at rounds 1 and 2. Since we know that the gossip scheme is valid, we conclude that $F_{k-1,n}$ can achieve gossiping in $k+1$ rounds for any even $n \leq 2F(k)-2$.

When $n = 2 \cdot F(k)$, we use the following gossip scheme in $F_{k,n}$:

- If k is even, that is $\vec{g}_n = k+1$ is odd, let each vertex communicate during round p along edges of dimension p for any $1 \leq p \leq k$. Then, let the vertices communicate again along dimension 1 during the round $k+1$.

- If k is odd (cf. Figure 2(a)), let each vertex communicate during round p along edges of dimension p for any $1 \leq p \leq k$. Then, let the vertices communicate again along dimension 2 during the round $k+1$.

First, we note that this gossip scheme also respects the 1-port hypothesis. To prove that $F_{k,n}$ is a gossip digraph, we use the same arguments as above. Indeed, Properties 3 and 4 can be extended to the k-th round, since here each vertex communicates at round $1 \leq p \leq k$ along dimension p. This means that if k is even (resp. odd), vertex $(1,0)$ has informed every vertex of the form $(1,j)$ (resp. $(2,j)$), with $0 \leq j \leq \frac{n}{2}-1$, after round k. Since for any k even (resp. odd), the last round takes place along dimension 1 (resp. 2), all the vertices of $F_{k,n}$ are informed after round $k+1$. Hence, vertex $(1,0)$ can broadcast in $F_{k,n}$ in $k+1$ rounds following the above scheme. As previously, we see that this is also true for any other vertex in $F_{k,n}$. Moreover, our gossip scheme respects the 1-port hypothesis, hence $F_{k,n}$ is a gossip digraph for any $n = 2F(k)$. □

Let us now present two methods allowing to get gossip digraphs from existing (minimum) gossip digraphs. They derive respectively from Properties 1 and 2. These methods prove to be rather efficient in providing upper bounds for $\vec{G}(n)$ in many cases.

Proposition 4 *For all $m \geq 0$, if $\vec{g}_{2m+2} = \vec{g}_{2m+1}$, then $\vec{G}(2m+2) \leq \vec{G}(2m+1)+2$.*

Proof : Let us consider a gossip digraph G of order $2m+1$. Since it is of odd order, we know that at round 1 (resp. $\vec{g}_{2m+1}$), one of its vertices v (resp. w) does not communicate. In that case, since $\vec{g}_{2m+1} = \vec{g}_{2m+2}$, it suffices to add a vertex u and two directed edges, namely (u,v) and (w,u) to get a gossip digraph of order $2m+2$. For this, use the same gossip scheme than in G, and let u communicate to v at round 1, and w communicate to u at round $\vec{g}_{2m+2}$. □

Proposition 5 *For all p and n such that $2^{i-1}n \leq p \leq 2^i n$ and $\vec{g}_p = \vec{g}_n + 2i$, we have $\vec{G}(p) \leq \vec{G}(n) + 2(p-n)$.*

Proof : The upper bound on $\vec{G}(p)$ follows directly from the recursive construction given in the Proof of Property 2. Indeed, we start with a gossip digraph of order n, having $\vec{G}(n)$ directed edges. Then we add $2n$ directed edges, and get a digraph of order $2n$ achieving gossiping in $\vec{g}_n + 2$ rounds. Then we apply the method recursively, and add $4n$, $8n$, etc. directed edges. Since we apply this method $(i-1)$ times, we get a digraph achieving gossiping in $\vec{g}_n + 2(i-1)$ rounds, and with $\vec{G}(n) + 2n + 4n + 8n + \ldots + 2^{i-1}n = \vec{G}(n) + 2^i n - 2n$ directed edges. The last step, as said in Proof of Property 2, consists in adding $(p - 2^{i-1}n)$ vertices and $2(p-2^{i-1}n)$ directed edges. Hence we get a digraph achieving gossiping in $\vec{g}_n + 2i$ rounds and with $\vec{G}(n) + 2(p-n)$ directed edges, which is a gossip digraph if $\vec{g}_p = \vec{g}_n + 2i$. □

Remark 1 *The above method is useful to get upper bounds for $\vec{G}(n)$, since for each added vertex, only two more directed edges are needed. In particular, it can improve significantly some of the upper bounds given in Theorem 5. For instance, a Corollary of Proposition 5 is the following : let $n = 2F(k-2)$; then, for all $n \leq p \leq 2n$ such that $\vec{g}_p = \vec{g}_n + 2 = k+1$*

(with $k \geq 4$), $\vec{G}(p) \leq 2p + (k-6) \cdot F(k-2)$. Indeed, we know by Theorem 1 that $\vec{g}_n = k-1$. Take $F_{k-2,n}$ as gossip digraph with n vertices. It holds $\frac{n(k-2)}{2}$ arcs. Now apply Proposition 5 with $i = 1$, and the result follows directly.

Note also that it is sometimes possible to save directed edges when using the method described above. Indeed, if we have the pattern shown in Figure 3 (left), we see that it can be replaced by the one shown on the right, without changing the ability of the digraph to gossip within $\vec{g}_p$ rounds.

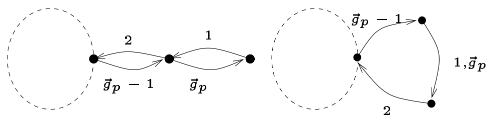

Figure 3: Saving one directed edge in a gossip digraph

5 Some gossip digraphs

5.1 Minimum gossip digraphs

Theorem 6 *$\vec{G}(2) = 2$, $\vec{G}(3) = 3$ and $\vec{G}(4) = 4$.*

Proof : First, $\vec{g}_2 = 2$ and $\vec{g}_4 = 4$ by Theorem 1. Moreover, Property 1 yields that $\vec{g}_4 \leq \vec{g}_3$, that is $\vec{g}_3 \geq 4$. Figure 4 shows that it is possible to gossip in a digraph with 3 vertices in 4 rounds, hence $\vec{g}_3 = 4$. Since we know that $\vec{G}(n) \geq n$ for any $n \geq 2$, the digraphs shown in Figure 4 are minimum gossip digraphs. □

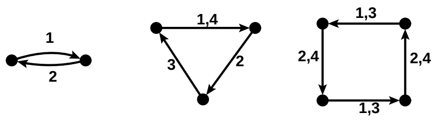

Figure 4: *MGD*s with 2, 3 and 4 vertices (left to right)

Theorem 7 *$\vec{g}_5 = 5$ and $\vec{G}(5) = 6$.*

Proof : Property 1 yields that for any $m \geq 0$, $\vec{g}_{2m+2} \leq \vec{g}_{2m+1}$. Moreover, Theorem 1 yields $\vec{g}_6 = 5$, since $2 \cdot F(4) = 6$. Hence $\vec{g}_5 \geq 5$. Suppose now that $\vec{g}_5 = 5$. We know that $\vec{G}(5) \geq 5$. Suppose $\vec{G}(5) = 5$. Then the corresponding digraph is the circuit, as shown in Figure 5(a). Let u_1 and u_5 be two vertices such that the directed distance $\vec{d}(u_1, u_5) = 4$; let u_2 be the successor of u_1 and u_3 be the successor of u_2. Then either u_1 communicates at round 1 with u_2 and u_2 communicates at round 2 with u_3, or the other way round, otherwise one of those vertices could not transmit its piece of information to the others. Suppose, w.l.o.g., that $1 \in (u_1, u_2)$ and $2 \in (u_2, u_3)$. Then, analogously, $1 \in (u_3, u_4)$ and $2 \in (u_4, u_5)$. Hence, u_5 cannot send its information before round 3, and therefore cannot reach u_4. Hence, if $\vec{g}_n = 5$, $\vec{G}(5) \geq 6$.

Moreover, the digraph shown in Figure 5(b) is a digraph with 5 vertices and 6 edges which achieves gossiping in 5 time units, hence the result. □

Theorem 8 *$\vec{G}(6) = 8$.*

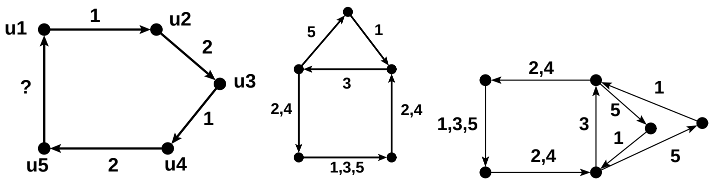

Figure 5: (a) A 5-circuit (b) A MGD with 5 vertices (c) A MGD with 6 vertices

Proof : Theorem 1 yields $\vec{g}_6 = 5$. The digraph shown in Figure 5(c) is a gossip digraph with 8 edges. Moreover, Proposition 2 yields $\vec{G}(6) \geq 7$. If we suppose $\vec{G}(6) = 7$ and detail the different possible cases, we show that it is impossible to gossip wihin 5 rounds (the proof is omitted here). Consequently, $\vec{G}(6) = 8$. □

Theorem 9 *$\vec{g}_7 = 6$ and $\vec{G}(7) = 9$.*

Proof : Even and Monien [EM89] showed that $\vec{g}_8 = 6$. Hence, thanks to Property 1, we know that $\vec{g}_7 \geq 6$. As Theorem 1 yields $4 \leq \vec{g}_7 \leq 6$, we immediately have $\vec{g}_7 = 6$. The three pairwise non-isomorphic digraphs shown in Figure 6 are gossip digraphs with 9 edges. Proposition 2 yields that $\vec{G}(7) \geq 8$. If we suppose $\vec{G}(7) = 8$ and detail the different cases, we show that it is impossible to gossip in 6 time units (the proof is omitted here). Hence the result. □

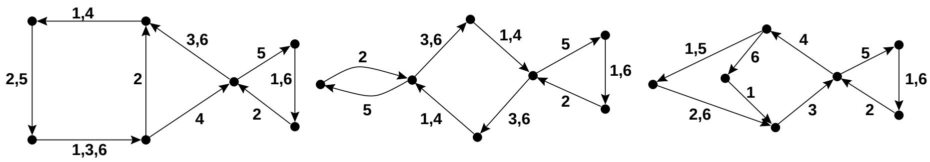

Figure 6: 3 MGDs with 7 vertices

Theorem 10 $\vec{G}(8) = 10$.

Proof : Even and Monien [EM89] showed that $\vec{g}_8 = 6$. Moreover, the digraph shown in Figure 7(a) is a gossip digraph with 8 vertices and 10 edges. Proposition 2 yields that $\vec{G}(8) \geq 9$. If we suppose $\vec{G}(8) = 9$ and detail the different cases, we show that it is impossible to gossip within 6 time units (the proof is omitted here). Hence $\vec{G}(8) = 10$. □

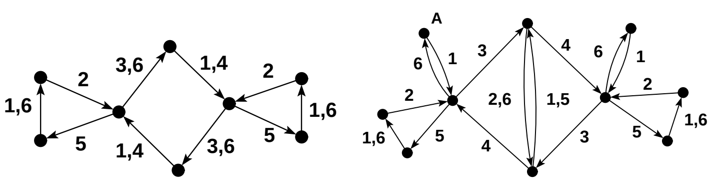

Figure 7: (a) A MGD with 8 vertices (b) A gossip digraph with 10 vertices

5.2 Gossip digraphs

In this section, we give different results concerning $\vec{g}_n$ and $\vec{G}(n)$. More precisely, we present here cases for which we show that the number of rounds is optimal, and some corresponding (but maybe not minimum) gossip digraphs.

Theorem 11 $\vec{g}_9 = 6$ *and* $11 \leq \vec{G}(9) \leq 14$; $12 \leq \vec{G}(10) \leq 16$.

Proof : Theorem 1 yields $\vec{g}_{10} = 6$ and Property 1 yields $\vec{g}_9 \geq 6$. Moreover, the digraph shown in Figure 7(b) holds 10 vertices and 16 edges, and it can achieve gossiping within 6 rounds. Hence $\vec{G}(10) \leq 16$. Moreover, if we suppress a vertex such as vertex A of Figure 7(b) and its incident directed edges, we get a digraph with 9 vertices that achieves gossiping in 6 rounds with the same gossip scheme.

Proposition 2 applied to the cases $n = 9$ and $n = 10$ gives respectively $\vec{G}(9) \geq 10$ and $\vec{G}(10) \geq 11$. However, if we suppose $\vec{G}(n) = n + 1$ for both $n = 9$ and $n = 10$, a case by case analysis shows that no digraph can be a gossip digraph (the proof is omitted here). Hence $\vec{G}(9) \geq 11$ and $\vec{G}(10) \geq 12$. □

Theorem 12 $\vec{g}_{13} = 7$ *and* $15 \leq \vec{G}(13) \leq 22$; $16 \leq \vec{G}(14) \leq 24$.

Proof : Theorem 3 yields $\vec{g}_{14} = 7$. Moreover, Property 1 yields $\vec{g}_{13} \geq 7$. The digraph shown in Figure 8 holds 14 vertices and 24 edges, and achieves gossiping in 7 time units. Hence $\vec{G}(14) \leq 24$. Moreover, if we suppress a vertex such as vertex A of Figure 8 and its two incident directed edges, we get a digraph with 13 vertices and 22 edges that achieves gossiping within 7 rounds too, using the same scheme. Hence $\vec{g}_{13} = 7$ and $\vec{G}(13) \leq 22$. The lower bound for $n = 13$ and $n = 14$ is given by Theorem 4. □

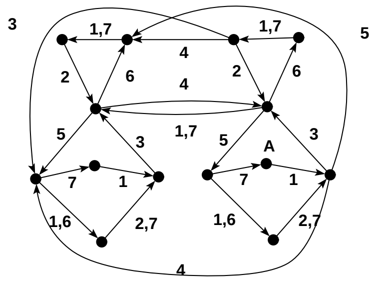

Figure 8: A gossip digraph with 14 vertices and its gossip scheme

5.3 Digraphs achieving gossiping in $k + 1$ or $k + 2$ rounds

In this section, we show several examples of digraphs with n vertices that achieve gossiping in $(k + 1)$ or $(k + 2)$ rounds, where k is the least integer such that $F(k) \geq \lfloor \frac{n}{2} \rfloor$. However, for the values of n we present here, we do not know whether the number of rounds indicated to achieve gossiping is optimal.

Theorem 13 *If* $\vec{g}_{11} = 7$, $\vec{G}(11) \leq 16$; *if* $\vec{g}_{12} = 7$, $\vec{G}(12) \leq 18$.

Proof : As seen previously, $\vec{g}_{11}$ as well as $\vec{g}_{12}$ are undetermined, though we know that $6 \leq \vec{g}_{12} \leq \vec{g}_{11} \leq 7$. The digraph shown in Figure 9(a) holds 12 vertices and achieves gossiping within 7 rounds. Hence, if $\vec{g}_{12} = 7$, $\vec{G}(12) \leq 18$. Moreover, if we suppress vertex A and its incident directed edges, we get a digraph with 11 vertices able to achieve gossiping in 7 rounds (following the same scheme). Hence, if $\vec{g}_{11} = 7$, $\vec{G}(11) \leq 16$.
Note that Figure 9(b) shows another digraph with 12 vertices and 18 edges achieving gossiping within 7 rounds, which is non isomorphic to the one of Figure 9(a). □

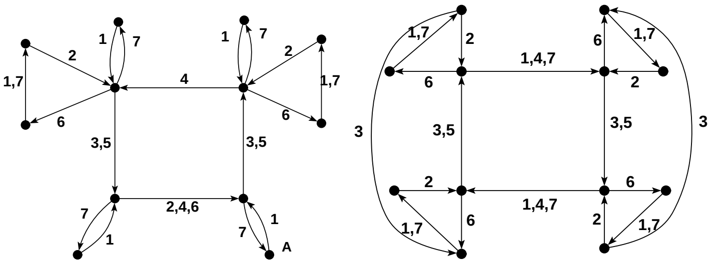

Figure 9: (a) and (b) Two digraphs with 12 vertices that achieve gossiping in 7 rounds

Theorem 14 *If* $\vec{g}_{15} = 8$, $\vec{G}(15) \leq 20$.

Proof : We know that $7 \leq \vec{g}_{15} \leq 8$. The digraph shown in Figure 10 achieves gossiping in 8 rounds and holds 20 edges. □

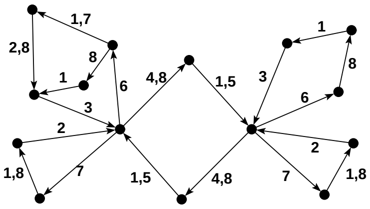

Figure 10: A digraph with 15 vertices such that $\vec{g}_{15} \leq 8$

Table 1 gives a summary of the results concerning $\vec{G}(n)$ for all $1 \leq n \leq 2 \cdot F(6)$, that is $1 \leq n \leq 16$. As the gossip times for some of those values is not precisely known, we give an upper and a lower bound for each of the hypotheses. To our knowledge, all these upper and lower bounds on $\vec{G}(n)$ are the best known so far.

n	$\vec{g}_n$	Lower	Upper	n	$\vec{g}_n$	Lower	Upper	$\vec{g}_n$	Lower	Upper
1	0	0	0*	**9**	6	11	14			
2	2	2	2*	**10**	6	12	16			
3	4	3	3*	**11**	6	13	30	7	12	16
4	4	4	4*	**12**	6	14	36	7	13	18
5	5	6	6*	**13**	7	15	22			
6	5	8	8*	**14**	7	16	24			
7	6	9	9*	**15**	7	17	49	8	17	20
8	6	10	10*	**16**	7	18	48			

Table 1: Bounds for $\vec{G}(n)$ $(1 \leq n \leq 16)$

6 Conclusion

This paper constitutes the first attempt to understand the structure of minimum gossip digraphs. However, we have been confronted to two main problems. The first one is the uncertainty concerning the optimal gossip time. Though the knowledge about the optimal gossip time has been improved by the different results of Section 2, determining precisely $\vec{g}_n$ for any n would clearly be a step forward. The second problem is the fact that the optimal time to gossip in the directed case is much "bigger" than in the undirected case ; this implies more difficulty in expressing strong constraints on the structure of minimum gossip digraphs. Consequently, the lower bounds on $\vec{G}(n)$ remain low.

However, it has been possible to determine the exact values of $\vec{G}(n)$ for all $n \leq 8$, and some close bounds have been given for several small values of n. Moreover, some more general properties, concerning either lower or upper bounds for $\vec{G}(n)$, have been shown.

References

[EM89] S. Even and B. Monien. On the number of rounds necessary to disseminate information. *Proc. 1st ACM Symp. on Parallel Algo. and Architectures*, pages 318–327, June 1989.

[ES79] R.C. Entringer and P.J. Slater. Gossips and telegraphs. *J. Franklin Institute*, 307:353–360, 1979.

[LRS97] A.L. Liestman, A. Raspaud, and T.C. Shermer. Private communication, 1997.

7 Annex : Complete Proofs of Theorems

7.1 Proof of Theorem 3

First, we recall the Theorem.

Theorem 15 *For all n, let k be the smallest integer such that $F(k) \geq \lfloor \frac{n}{2} \rfloor$. Let $\rho = \frac{1+\sqrt{5}}{2}$ and $\overline{\rho} = 1 - \rho = \frac{1+\sqrt{5}}{2}$. Then :*

(a) If n and k are even and $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1)$, then $\vec{g}_n = k+1$;

(b) If n is even, k is odd and $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) + \frac{2}{2\rho+1} \cdot (-\overline{\rho})^k$, then $\vec{g}_n = k+1$;

(c) If n is odd, k is even and $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) - 1$, then $\vec{g}_n = k+1$ or $\vec{g}_n = k+2$;

(d) If n and k are odd and $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) + \frac{2}{2\rho+1} \cdot (-\overline{\rho})^k - 1$, then $\vec{g}_n = k+1$ or $\vec{g}_n = k+2$.

Proof : The aim is to show that, in each case, for such values of n, $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, and the result follows directly by Theorems 2 and 1. This is straightforward thanks to the following equalities and inequalities : $F(k) = \frac{\rho^k - (\overline{\rho})^k}{\sqrt{5}}$, $2\rho + 1 = \rho^3$ and $-1 < \overline{\rho} < 0$.

The proof, in each of the four cases, relies on the same arguments. Let us distinguish all the cases, and discuss the correctness of the Theorem :

- n is even, k is even : in that case, $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1)$ becomes $\frac{n}{2} \geq \frac{\rho^{k+1} - (\overline{\rho})^{k+1}}{\rho^3}$ by the first two equalities. Since k is even, we have $(\overline{\rho})^{k+1} < 0$, that is $\frac{n}{2} > \rho^{k-2}$. Hence $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, which, by Theorems 1 and 2, yields $\vec{g}_n = k+1$.

- n is even, k is odd : in that case, $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) + \frac{2}{2\rho+1} \cdot (-\overline{\rho})^k$ becomes $\frac{n}{2} \geq \frac{\rho^{k+1} - (\overline{\rho})^{k+1} + (-\overline{\rho})^k}{\rho^3}$ by the first two equalities, that is $\frac{n}{2} \geq \frac{\rho^{k+1} - (\overline{\rho})^k \cdot (\overline{\rho}+1)}{\rho^3}$. Since k is odd and $-1 < \overline{\rho} < 0$, we have $(\overline{\rho})^k < 0$ and $\overline{\rho} + 1 > 0$, that is $\frac{n}{2} > \rho^{k-2}$. Hence $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, which, by Theorems 1 and 2, yields $\vec{g}_n = k+1$.

- n is odd, k is even : in that case, $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) - 1$ becomes $\frac{n+1}{2} \geq \frac{\rho^{k+1} - (\overline{\rho})^{k+1}}{\rho^3}$ by the first two equalities. Since k is even, we have $(\overline{\rho})^{k+1} < 0$, that is $\frac{n+1}{2} > \rho^{k-2}$. Hence $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, which, by Theorems 1 and 2, yields $\vec{g}_n > k$, that is $\vec{g}_n = k+1$ or $\vec{g}_n = k+2$.

- n is odd, k is odd : in that case, $n \geq \frac{2\sqrt{5}}{2\rho+1} \cdot F(k+1) + \frac{2}{2\rho+1} \cdot (-\overline{\rho})^k - 1$ becomes $\frac{n+1}{2} \geq \frac{\rho^{k+1} - (\overline{\rho})^k \cdot (\overline{\rho}+1)}{\rho^3}$ by the first two equalities. Since k is odd, we have $(\overline{\rho})^k \cdot (\overline{\rho}+1) < 0$, that is $\frac{n+1}{2} > \rho^{k-2}$. Hence $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, which, by Theorems 1 and 2, yields $\vec{g}_n > k$, that is $\vec{g}_n = k+1$ or $\vec{g}_n = k+2$.

Standard calculations then yield that, in every case, $2 + \lceil log_\rho \lceil \frac{n}{2} \rceil \rceil > k$, which implies, by Theorems 1 and 2, $\vec{g}_n \geq k+1$. □

7.2 Proof of Theorems 8 to 11

In this Section, the proof, for each of the Theorems, consists in a (somewhat) tedious case by case analysis, which is provided below. Before proving the Theorems, let us note the following.

Property 5 *Let G be a gossip digraph, and let i be the number of adjacent vertices of degree 2 in G. We have :*

- *$i \leq i_m$, where i_m is the greatest integer such that $i_m + 2^{\vec{g_n} - i_m} \geq n + 1$;*
- *G must be strongly connected.*

Proof : The first Property has been discussed in Proof of Theorem 4, while the second is straightforward : indeed, if the digraph is not strongly connected, then there exists a vertex u which cannot send its information to all the others vertices of G. □

7.2.1 Proof of Theorem 8

We know that $\vec{g_6} = 5$ and $\vec{G}(6) \geq 7$. We also know, thanks to Property 5, that there cannot be more that $i_m = 3$ adjacent vertices of degree 2. Now suppose $\vec{G}(6) = 7$. Since we know that each vertex u must verify $d^+(u) \geq 1$ (resp. $d^-(u) \geq 1$), we get two cases for the vertices degrees : either two of the 6 vertices are of degree 3, or one of them is of degree 4, all the other vertices being of degree 2. Let us detail those two cases :

- 1 vertex is of degree 4. Let us then consider the undirected graphs resulting from this case. There is only one such graph (cf. Figure 11 (a)). Indeed, let u be the vertex of degree 4. One can see easily that u must be such that $d^+(u) = d^-(u) = 2$, otherwise at least one vertex would not be able to broadcast its information in the digraph. Moreover, there are two cycles containing u, and these cycles must be circuits in the digraph, otherwise some vertices could not inform or be informed. Hence there is only one such digraph G, as shown in Figure 11 (b).
Now, let us show that G is not a *MGD*. We see that $\vec{d}(u_1, v_1) = \vec{g_6} = 5$, where $\vec{d}(u_1, v_1)$ is the directed distance between vertex u_1 and vertex v_1. Hence there must be a communication at round 3 along the directed edge (v_2, u). But $\vec{d}(u_2, v_2) = 5$ too, and consequently there must be a communication at round 3 along the directed edge (u, u_1). This is not possible due to the 1-port model. Hence G is not a gossip digraph.

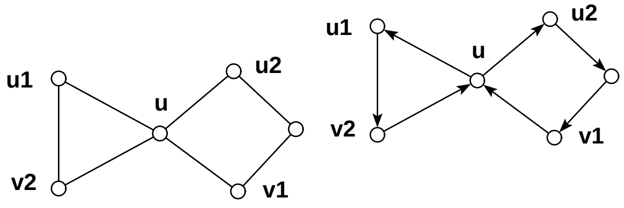

Figure 11: (a) The only graph corresponding (b) The only digraph corresponding

- 2 vertices are of degree 3 : in that case, there are more subcases to consider. Let u and v be those two vertices. Necessarily, there must be at least two directed paths between u and v (in opposite directions), otherwise we could not achieve gossiping in the digraph (cf. Property 5). Hence, if we build the corresponding undirected graph, we get three cases, as shown in Figure 12. For each of those cases, we will consider all the possible directed graphs that can be obtained, and will show that they cannot be gossip digraphs. The idea to prove this is always the same, and is as follows : we find a couple of vertices x_1 and x_2 such that there is only one directed path to go from x_1 to x_2, and such that the directed distance $\vec{d}(x_1, x_2) = \vec{g}_n$. Hence, this forces the broadcast from x_1 to x_2 to take place along this directed path, and to use rounds $1, 2, \ldots, \vec{g}_n$. Then, we find another couple of vertices y_1 and y_2 for which the broadcast from y_1 to y_2 is impossible within $\vec{g}_n$ rounds, due to the broadcast from x_1 to x_2 and the fact that we are in the 1-port model.

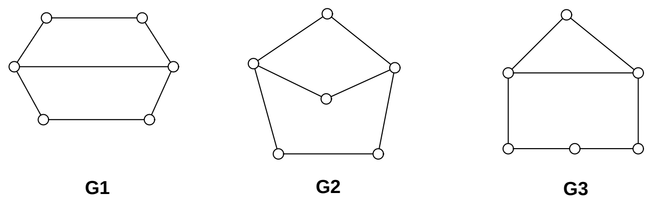

Figure 12: 3 graphs according to the hypothesis

(1) Graph G_1 : there are two ways to give an orientation to the graph, as shown in Figure 13. In both cases, we see that gossiping cannot be achieved in 5 rounds in any of those digraphs. If we consider the left digraph of Figure 13, we have $\vec{d}(u_1, u) = 5$ and $\vec{d}(u_2, u_1) = 5$, which contradicts the 1-port hypothesis. Concerning the right digraph of Figure 13, we have $\vec{d}(v_2, u_2) = 5$ and $\vec{d}(u_1, v_1) = 5$, which leads to the same kind of contradiction.

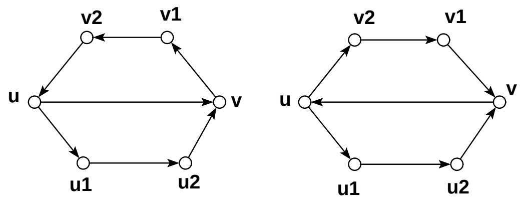

Figure 13: 2 possible digraphs for graph G_1

(2) Graph G_2 : there are two ways to give an orientation to the graph, as shown in Figure 14. In both cases, we see that gossiping cannot be achieved in 5 rounds in any of those digraphs. Let us consider the left digraph of Figure 14. We have $\vec{d}(u_1, u') = 5$ and $\vec{d}(u', u_2) = 5$, which leads to a contradiction. Similarly, in the right digraph of Figure 14, we have $\vec{d}(u', v') = 5$ and $\vec{d}(v', u') = 5$.

(3) Graph G_3 : there are three ways to give an orientation to the graph, as shown in Figure 15. In all the cases, we see that gossiping cannot be achieved in 5 rounds in any of those digraphs. Let us consider first the left digraph of Figure 15. We have $\vec{d}(u_1, u) = 5$ and $\vec{d}(u_2, u_1) = 5$, which leads to a contradiction. Similarly, in

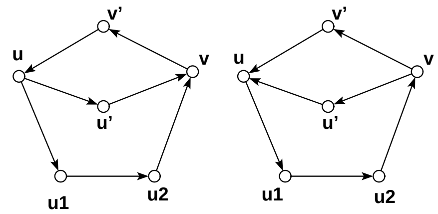

Figure 14: 2 possible digraphs for graph G_2

the middle digraph of Figure 15, we have $\vec{d}(u_1, u') = 5$, but in that case u_2 cannot broadcast to u_1 within 5 rounds. Finally, in the right digraph of Figure 15, we have $\vec{d}(u', v) = 5$ and in that case v is unable to broadcast to u_3 within 5 rounds.

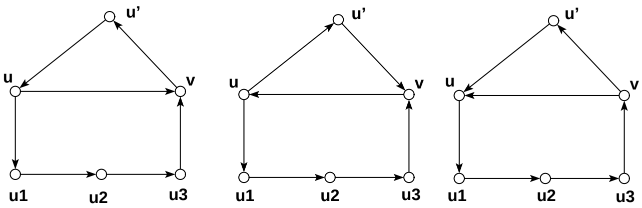

Figure 15: 3 possible digraphs for graph G_3

We then see that the assumption $\vec{G}(6) = 7$ gives digraphs which are not able to gossip within $\vec{g}_6$ rounds. Hence $\vec{G}(6) \geq 8$. Since we know that $\vec{G}(6) \leq 8$ (cf. Figure 5(c)), we get directly Theorem 8.

7.2.2 Proof of Theorem 9

We will prove the Theorem exactly the same way than in the previous Section. In this case, we have $\vec{g}_7 = 6$ and $\vec{G}(7) \geq 8$. Let us now suppose $\vec{G}(7) = 8$ and show that this cannot give us a gossip digraph. By Property 5, we know that there cannot be more than 4 adjacent vertices of degree 2.

As seen before, we have two cases : either one vertex is of degree 4, or two are of degree 3.

- 1 vertex is of degree 4 : in that case, there are two possibilities concerning the corresponding digraph. This is due to the fact that $i_m = 4$ for our purpose. The two digraphs are displayed in Figure 16. Let us then show that they cannot be gossip digraphs. Concerning Figure 16 (a), we have $\vec{d}(u_1, u_6) = \vec{d}(u_3, u_2) = 6$. Hence the scheme displayed in Figure 16 (a). But, starting from this point, we see that it is impossible for u_5 to send its information to u_4 within 6 rounds.
  Concerning the digraph displayed in Figure 16 (b), we see that $\vec{d}(u_1, u_2) = 6$ and $\vec{d}(v_1, v_2) = 6$, which leads to a contradiction.

- 2 vertices are of degree 3 : in that case, the (undirected) graphs which could correspond to this hypothesis are displayed in Figure 17. For each of those graphs, let us detail the possible ways to direct them, and discuss whether the digraphs obtained are gossip digraphs.

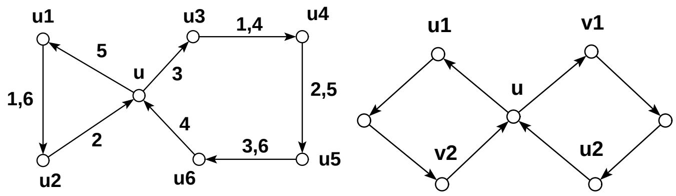

Figure 16: (a) and (b) The 2 possible digraphs according to the hypothesis

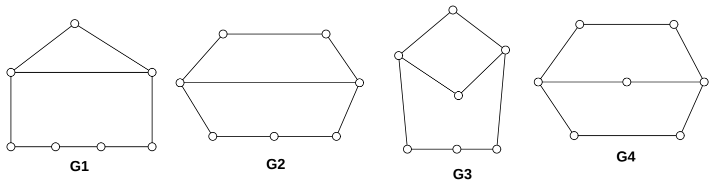

Figure 17: 4 graphs according to the hypothesis

(1) Graph G_1 : there are three possible digraphs in that case (cf. for this Figure 18). The left digraph of Figure 18 cannot be a gossip digraph, since $\vec{d}(u', u_4) = 6$, $\vec{d}(u_1, u') = 6$ and consequently there is no way u_2 can broadcast to u_1 within 6 rounds, due to the 1-port model. Similarly, the middle digraph of Figure 18 is not a gossip digraph, since $\vec{d}(u_1, u) = 6$ and $\vec{d}(u_2, u_1) = 6$. Finally, the same goes for the right digraph of Figure 18, due to the fact that $\vec{d}(u', v) = 6$ and $\vec{d}(u, u') = 6$.

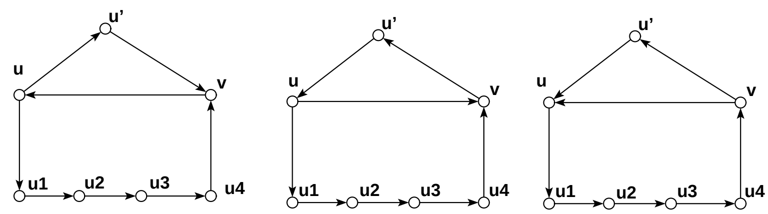

Figure 18: 3 possible digraphs for G_1

(2) Graph G_2 : there are three possible digraphs in that case (cf. for this Figure 19). The left digraph of Figure 19 cannot be a gossip digraph, since $\vec{d}(u', u_3) = 6$ and $\vec{d}(u_1, v') = 6$. Similarly, the middle digraph of Figure 19 is not a gossip digraph, since $\vec{d}(u_1, u) = 6$ and $\vec{d}(u_2, u_1) = 6$. Finally, the same goes for the right digraph of Figure 18, due to the fact that $\vec{d}(v', v) = 6$ and $\vec{d}(u', v') = 6$.

(3) Graph G_3 : there are two possible digraphs in that case (cf. for this Figure 20). The left digraph of Figure 20 cannot be a gossip digraph, since $\vec{d}(u_1, u') = 6$, and consequently there is no way u_2 can broadcast to u_1 within 6 rounds, due to the 1-port model. The same goes for the right digraph of Figure 20, due to the fact that $\vec{d}(u', v') = 6$ and $\vec{d}(v', u') = 6$.

(4) Graph G_4 : there are two possible digraphs in that case (cf. for this Figure 21). The left digraph of Figure 21 cannot be a gossip digraph, since $\vec{d}(w, v') = 6$, and $\vec{d}(u', w) = 6$. The same goes for the right digraph of Figure 21, due to the fact that $\vec{d}(u_2, u') = 6$ and $\vec{d}(v', u_1) = 6$.

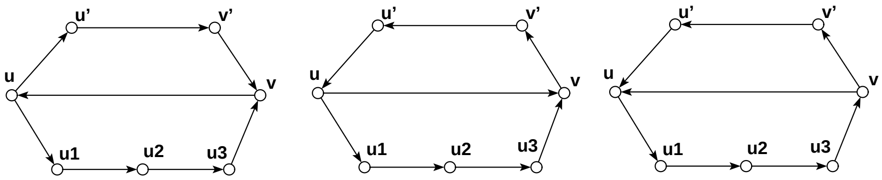

Figure 19: 3 possible digraphs for G_2

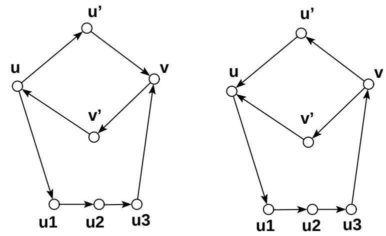

Figure 20: 2 possible digraphs for G_3

Hence, we see that the assumption $\vec{G}(7) = 8$ gives digraphs which are not able to gossip within $\vec{g}_7$ rounds. Hence $\vec{G}(7) \geq 9$. Since we know that $\vec{G}(7) \leq 9$ (cf. Figure 6), we get directly Theorem 9.

7.2.3 Proof of Theorem 10

The method here remains the same as before. In this case, we have $\vec{g}_8 = 6$ and $\vec{G}(8) \geq 9$. Let us now suppose $\vec{G}(8) = 9$ and show that this cannot give us a gossip digraph. By Property 5, we know that $i_m = 3$, i.e. there cannot be more than 3 adjacent vertices of degree 2. As seen before, we have two cases : either one vertex is of degree 4, or two are of degree 3.

- 1 vertex is of degree 4 : we see that this case is impossible due to the fact that $i_m = 3$.
- 2 vertices are of degree 3 : still due to the fact that $i_m = 3$, this leads to three possible graphs, in the undirected case (cf. Figure 22). As before, we will detail, for each graph, the different possibilities concerning the corresponding digraphs, and we will discuss whether those digraphs are gossip digraphs.

  (1) Graph G_1 : there are two possible digraphs in that case (cf. for this Figure 23). The left digraph of Figure 24 cannot be a gossip digraph, since $\vec{d}(u_1, u) = 7$. Similarly, the right digraph of Figure 24 is not a gossip digraph, since $\vec{d}(u', u_3) = 7$.

  (2) Graph G_2 : there are three possible digraphs in that case (cf. for this Figure 24). The left digraph of Figure 23 cannot be a gossip digraph, since $\vec{d}(u_1, u') = 7$.

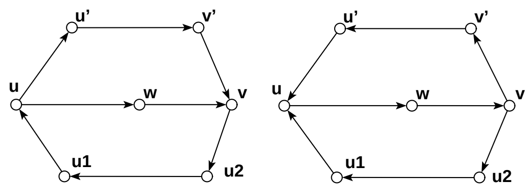

Figure 21: 2 possible digraphs for G_4

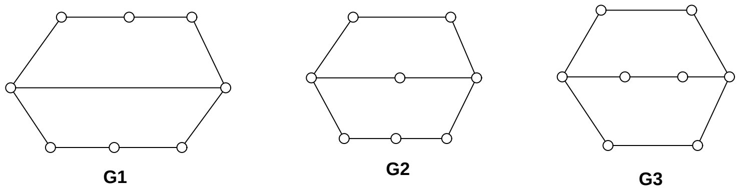

Figure 22: 3 graphs according to the hypothesis

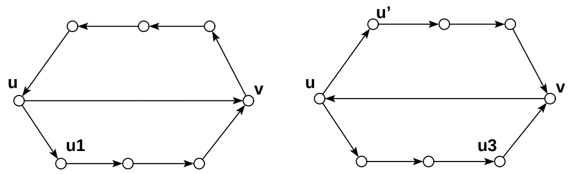

Figure 23: 3 possible digraphs for G_1

Similarly, the middle digraph of Figure 23 is not a gossip digraph, since $\vec{d}(u_1, v_1) = 7$. Finally, the same goes for the right digraph of Figure 23, due to the fact that $\vec{d}(v_1, u') = 7$.

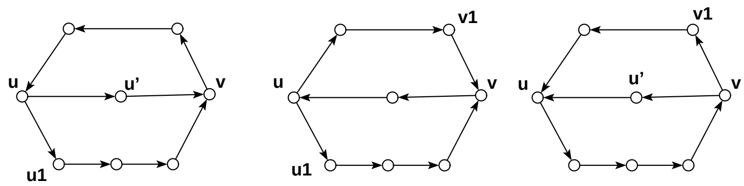

Figure 24: 3 possible digraphs for G_2

(3) Graph G_3 : there is only one possible digraph in that case (cf. for this Figure 25). This digraph cannot be a gossip digraph, since $\vec{d}(u_1, v_2) = 7$.

Hence, we see that the assumption $\vec{G}(8) = 9$ gives digraphs which are not able to gossip within $\vec{g}_8$ rounds. Hence $\vec{G}(8) \geq 10$. Since we know that $\vec{G}(8) \leq 10$ (cf. Figure 7), we get directly Theorem 10.

7.2.4 Proof of Theorem 11

In this Section, we prove the lower bounds for $\vec{G}(9)$ and $\vec{G}(10)$ given in Theorem 11. The proof relies on similar arguments than above.

$\vec{G}(9) \geq 11$ We already know that $\vec{G}(9) \geq 10$ by Proposition 2. We also know that i_m, the maximum number of adjacent vertices of degree 2 in a MGD_9, is equal to 3 (by Property 5). Now, if we suppose $\vec{G}(9) = 10$, then we have, as seen before, two cases.

- 1 vertex is of degree 4 : in that case, we see that this is impossible since $i_m = 3$.
- 2 vertices are of degree 3 : in that case, this leads to 2 (undirected) graphs, as shown in Figure 26. Now, let us detail the possible orientations for each of these graphs, and discuss whether these digraphs are gossip digraphs.

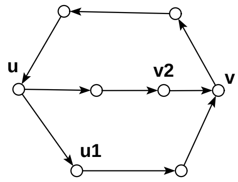

Figure 25: The only possible digraph for G_3

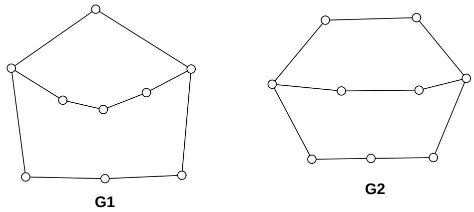

Figure 26: 2 graphs according to the hypothesis

(1) Graph G_1 : there are two possible digraphs in that case, as shown in Figure 27. The left digraph of Figure 27 cannot be a gossip digraph, since $\vec{d}(u_1', u_3) = 8$. The same goes for the right digraph of Figure 27, since $\vec{d}(u_1, u') = 8$.

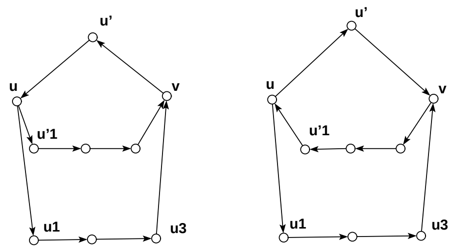

Figure 27: 2 possible digraphs for G_1

(2) Graph G_2 : there are two possible digraphs in that case, as shown in Figure 28. The left digraph of Figure 28 cannot be a gossip digraph, since $\vec{d}(u_1, u_3') = 8$. The same goes for the right digraph of Figure 27, since $\vec{d}(v', u_1') = 8$.

Hence we see that $\vec{G}(9) \neq 10$.

$\vec{G}(10) \geq 12$ We already know that $\vec{G}(10) \geq 11$ by Proposition 2. We also know that i_m, the maximum number of adjacent vertices of degree 2 in a MGD_{10}, is equal to 3 (by Property 5). Now, if we suppose $\vec{G}(10) = 11$, then we have, as seen before, two cases.

- 1 vertex is of degree 4 : in that case, we see that this is impossible since $i_m = 3$.
- 2 vertices are of degree 3 : in that case, this leads to 1 (undirected) graph, as shown in Figure 29(a). Now, let us detail the possible orientations for this graph, and discuss whether these digraphs are gossip digraphs.
  There are two possible digraphs, as shown in Figure 29(b). The middle digraph of Figure 29 cannot be a gossip digraph, since $\vec{d}(u_1, u_3') = 9$. The same goes for the right digraph of Figure 29, since $\vec{d}(u_3', u_1'') = 9$.

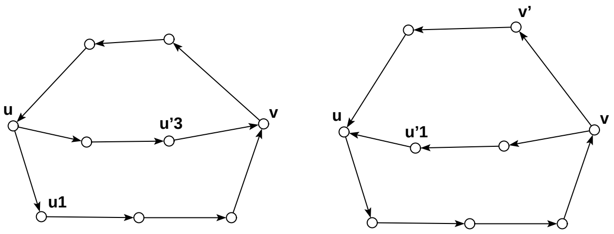

Figure 28: 2 possible digraphs for G_2

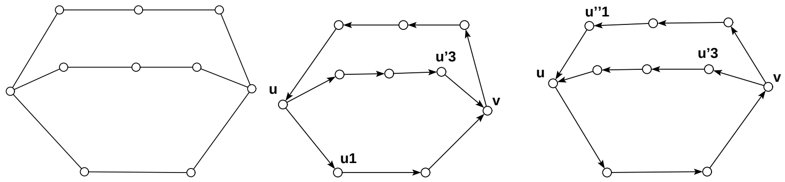

Figure 29: (a) The only corresponding graph (b) The 2 possible corresponding digraphs

Hence we see that $\vec{G}(10) \neq 11$.

7.3 Other Results

In this Section, we give some more results on $\vec{G}(n)$ for $17 \leq n \leq 26 = 2 \cdot F(7)$. Among other things, these results, along with the ones of Table 1 show that for any $5 \leq n \leq 22$, $\vec{g}_n \leq n-1$ (cf. Proof of Observation 1).

Theorem 16 *If $\vec{g}_n = 8$ for the following values of n, then $\vec{G}(17) \leq 24$, $\vec{G}(18) \leq 26$, $\vec{G}(19) \leq 28$ and $\vec{G}(20) \leq 30$.*

Proof : Consider the digraph shown in Figure 30(a). This digraph holds 20 vertices and 30 edges and achieves gossiping within 8 rounds. Hence if $\vec{g}_{20} = 8$, $\vec{G}(20) \leq 30$. Moreover, it remains possible to delete some vertices (namely A_1, A_2 and A_3) and their adjacent directed edges without affecting the ability of the graph to gossip within 8 rounds. Hence $G(\vec{1}9) \leq 28$, $G(\vec{1}8) \leq 26$ and $\vec{G}(17) \leq 24$, provided that $\vec{g}_n = 8$ for those values of n. □

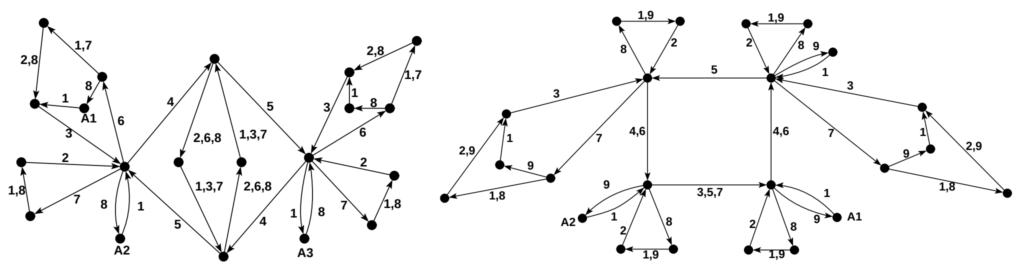

Figure 30: (a) A digraph with 20 vertices s.t. $\vec{g}_{20} \leq 8$ (b) A digraph with 23 vertices s.t. $\vec{g}_{23} \leq 9$

Theorem 17 *If $\vec{g}_n = 9$ for $n = 21$ and $n = 23$, then $\vec{G}(21) \leq 30$ and $\vec{G}(23) \leq 34$.*

Proof : The digraph shown in Figure 30(b) holds 23 vertices and 34 edges and is able to gossip within 9 rounds. Hence $\vec{G}(23) \leq 34$ if $\vec{g}_{23} = 9$. Moreover, it is possible to delete 2 vertices and 4 edges (for instance, A_1, A_2 and their incident edges) without affecting the ability of the digraph to gossip within 9 rounds. Hence, $\vec{G}(21) \leq 30$ if $\vec{g}_{21} = 9$. □

n	$\vec{g}_n$	Lower	Upper	$\vec{g}_n$	Lower	Upper	$\vec{g}_n$	Lower	Upper
17	7	19	56	8	19	24			
18	7	20	63	8	20	26			
19	7	22	63	8	21	28			
20	7	23	70	8	22	30			
21	7	24	70	8	23	80	9	23	30
22	7	25	77	8	24	66			
23	8	25	88	9	25	34			
24	8	26	72						
25	8	27	96	9	27	40			
26	8	28	91						

Table 2: Bounds for $\vec{G}(n)$ $(1 \leq n \leq 26)$