On the Structure of Minimum Broadcast Digraphs

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Abstract

Broadcasting is a problem of information dissemination described in a group of individuals connected by a communication network, where one individual has an item of information and needs to communicate it to everyone else. This communication pattern finds its main applications in the field of interconnection networks for parallel and distributed architecture. Numerous previous papers have investigated ways to construct sparse undirected graphs (networks) in which this process can be completed in minimum time. In this paper, we consider the broadcast problem in directed graphs. We describe some techniques to construct sparse digraphs on n vertices in which broadcasting can be completed in minimum time. For $n = 2^p - 1$ *and* $n = 2^p - 2$*, we show that these techniques produce the sparsest possible digraphs of this type (called minimum broadcast digraphs, or* MBD*s). In the case* $n = 2^p - 1$*, we give one class of* MBD*s, as for the case* $n = 2^p - 2$*, we give two non isomorphic classes of* MBD*s. We show that these techniques also produce a class of* MBD*s on* $n = 2^p$ *vertices which is non isomorphic to the one given in [LP92]. For some other infinite classes of values of n, we give techniques that produce the sparsest known digraphs of this type, and we also give some lower bounds on the size of* MBD*s. Finally, in the range* $1..32$*, we give new* MBD*s which are not isomorphic to the ones given in [LP92] (namely* $n = 6$*,* $n = 9$*,* $n = 14$ *and* $n = 30$*).*

1 Introduction

Broadcasting refers to the process of dissemination of information in a communication network where a message, originated by one member, has to be transmitted to all the other members of the network. This is achieved by placing communication calls over the communication lines of the network. We will consider a *constant-time*, *1-port* model, that is each call requires one unit of time and a vertex can participate in only one call per unit, provided that a vertex can only call a vertex to which it is adjacent. Given a strongly connected digraph G, $\vec{b}(G)$ will denote the amount of time necessary to broadcast in G from any vertex v of G, or the *broadcast time* of G. If we consider the complete digraph K_n^* of order n, we can easily see that $\vec{b}(K_n^*) = \lceil log_2(n) \rceil$, since the number of informed vertices can at most double every time unit. Let $\vec{b_n}$ be this value of $\vec{b}(K_n^*)$. A *broadcast digraph* will denote any digraph able to broadcast in minimum time. However, it is not necessary to consider the complete digraph K_n^* to get a broadcast digraph. We then call a *Minimum Broadcast Digraph*, or MBD, any broadcast digraph with a minimum number of directed edges. This number will be denoted by $\vec{B}(n)$.

From an application perspective, MBDs represent the cheapest possible communication networks (e.g. with a minimum number of communication lines) in which broadcasting can be achieved from any vertex in minimum time.

Analogous definitions have been previously given for undirected graphs (cf. [HHL88]) : the broadcast time of a vertex v in a graph G will be denoted by $b(v)$, and the number of edges of a *minimum broadcast graph*, or MBG, is denoted by $B(n)$.

This paper is organized as follows : Section 2 will recall some known general results given in [LP92] and [PC94]. Section 3 will be devoted to new general results on $\vec{B}(n)$, while in Section 4 we will present some particular cases improving the bounds given in [LP92], as well as some new MBDs for small values of n.

2 Known results

In this section, we intend to recall general results about $\vec{B}(n)$ for infinite classes of values of n. However, many particular cases have been sorted out in [LP92], that we will not recall here. We refer to [LP92] and [PC94] for a more detailed information about the structure of MBDs.

In [PC94], however, the aim of the study is not to find MBDs. Their goal was to find minimal broadcast digraphs (that is, broadcast digraphs with "few" edges) that have the property of being regular. Those digraphs, from our point of view, will consequently give us upper bounds for $\vec{B}(n)$. In particular, Park and Chwa build a class of circulant digraphs and show that they are regular minimal broadcast digraphs.

Definition 1 *A* circulant digraph *on n vertices $C'_n(a_1, a_2, \ldots, a_p)$, $a_1 < a_2 < \ldots < a_p$, has vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E = \{(v_x, v_y) \mid \exists\ a_i,\ 1 \leq i \leq p$ such that $x + a_i \equiv y \pmod{n}\}$.*

Park and Chwa showed that $C'_n(2^1 - 1, 2^2 - 1, \ldots, 2^{\lfloor log_2 n \rfloor} - 1)$ is a minimal broadcast digraph for any n. Moreover, such a digraph is $\lfloor log_2 n \rfloor$-regular. This theorem can be transformed, from our point of view, into a general upper bound for $\vec{B}(n)$. Indeed, if n is not a power of 2, $\lfloor log_2 n \rfloor = \vec{b}_n - 1$, where $\vec{b}_n$ is the broadcast time. Hence the following theorem :

Theorem 1 *For all $2^{p-1} + 1 \leq n \leq 2^p - 1$, $\vec{B}(n) \leq n \times (p-1)$.*

Moreover, Park and Chwa [PC94] showed the following theorems :

Theorem 2 *For all $2^{p-1} + 1 \leq n \leq 2^{p-1} + 2^{p-2}$ with $p \geq 4$, there exists a regular minimal broadcast digraph of order n and regular of degree $\lfloor log_2 n \rfloor - 1$.*

Theorem 3 *For all $2^{p-1} + 1 \leq n \leq 2^{p-1} + 2^{p-4}$ with $p \geq 5$, there exists a regular minimal broadcast digraph of order n and regular of degree $\lfloor log_2 n \rfloor - 2$.*

Those theorems can be translated, from our point of view, to the following ones :

Theorem 4 *For all $2^{p-1} + 1 \leq n \leq 2^{p-1} + 2^{p-2}$ with $p \geq 4$, $\vec{B}(n) \leq n \times (p-2)$.*

Theorem 5 *For all $2^{p-1} + 1 \leq n \leq 2^{p-1} + 2^{p-4}$ with $p \geq 5$, $\vec{B}(n) \leq n \times (p-3)$.*

Finally, Liestman and Peters [LP92] have shown the following theorem, which is the only exact known general value of $\vec{B}(n)$ for an infinite class of values of n.

Theorem 6 *$\vec{B}(2^p) = p \times 2^p$.*

Proof : It is not difficult to see that any vertex of outdegree strictly less than p cannot inform n vertices in minimum time. Moreover, we can take any (undirected) MBG on 2^p vertices and replace each edge with a pair of symmetric directed edges to get a broadcast digraph, hence the result. Note that in that case any MBD built that way is such that any of its vertices has p neighbours.

3 New results

3.1 A new class of MBDs of order 2^p

Theorem 7 *The family of circulant digraphs $C'_{2^p}(1, 3, \ldots, 2^p - 1)$ $(p \geq 3)$ is a class of MBDs on 2^p vertices non isomorphic to the one given in Theorem 6.*

Proof : First, it is not difficult to see that $C'_{2^p}(1, 3, \ldots, 2^p - 1)$ is a MBD for $n = 2^p$, since in that case $\lfloor log_2 n \rfloor = \lceil log_2 n \rceil = p$, and consequently such a digraph is of minimum size for broadcasting. We know that each vertex of any MBD built as in proof of Theorem 6 has p neighbours. By definition, in $C'_{2^p}(1, 3, \ldots, 2^p - 1)$, each vertex has at least p neighbours. Moreover, if each vertex v_i had exactly p neighbours, there would be a directed edge $v_i v_j$ and a directed edge $v_j v_i$ for each neighbour v_j of v_i. In particular, let $v_i = v_0$ and $v_j = v_3$. By definition of $C'_{2^p}(1, 3, \ldots, 2^p - 1)$, there would be a k such that $3 + 2^k - 1 \equiv 0 \bmod n$, that is $2^k + 2 = 2^p$. This is only possible for $p = 2$ and $k = 1$. Hence in $C'_{2^p}(1, 3, \ldots, 2^p - 1)$ with $p \geq 3$, any vertex has at least $(p + 1)$ neighbours. Consequently, for any $p \geq 3$, $C'_{2^p}(1, 3, \ldots, 2^p - 1)$ and the MBD given in [LP92] are non isomorphic.

3.2 Exact values of $\vec{B}(2^p - 1)$ and $\vec{B}(2^p - 2)$

Theorem 8 *For all $p \geq 3$:*

- $\vec{B}(2^p - 2) = (p - 1) \times (2^p - 2)$;
- $\vec{B}(2^p - 1) = (p - 1) \times (2^p - 1)$.

Proof : In both cases, that is $n = 2^p - 1$ and $n = 2^p - 2$, it is not difficult to see that any vertex of outdegree strictly less than $(p - 1)$ cannot inform more than $2^p - 3$ vertices on the whole within p time units. Hence $\vec{B}(n) \geq n \times (p - 1)$. Moreover, the result given by Park and Chwa, that was transformed into Theorem 1 in the previous section, yields $\vec{B}(n) \leq n \times (p - 1)$; hence the result. Consequently, for any $n = 2^p - 1$ or $n = 2^p - 2$, $C'_n(1, 3, 7, \ldots, 2^{\lfloor log_2 n \rfloor} - 1)$ is a MBD.

3.3 A second class of MBDs for $n = 2^p - 2$

We have seen that the circulant digraphs $C'_n(1, 3, \ldots, 2^{\lfloor log_2 n \rfloor} - 1)$ were MBDs for $n = 2^p - 1$ and $n = 2^p - 2$. However, there is a second class of MBDs for $n = 2^p - 2$ which is non isomorphic to the circulant digraphs defined above for any $p \geq 3$. They are what we can call the *Knödel digraphs*. Below is a definition of the Knödel graphs in the undirected case.

Definition 2 *The* Knödel graph *[FP94] on $n \geq 2$ vertices (n even) and of maximum degree $\Delta \geq 1$ is denoted $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the couples (i, j) with i=1,2 and $0 \leq j \leq \frac{n}{2} - 1$. For every j, $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, j + 2^k - 1 \bmod \frac{n}{2})$, for $k = 0, \ldots, \Delta - 1$.*

For $0 \leq k \leq \Delta - 1$, an edge of $W_{\Delta,n}$ which connects a vertex $(1, j)$ to the vertex $(2, j + 2^k - 1 \bmod \frac{n}{2})$ is said to be *in dimension k*.

It has been shown in [Fer97] that $W_{p-1,n}$ is a gossip graph (hence a broadcast graph) for any even n not a power of 2 and $p = \lceil log_2 n \rceil$. It suffices for any vertex u to communicate at time $1 \leq t \leq p - 1$ along dimension $(t - 1)$, and, during the last time unit, to communicate again along dimension 0.

Now let a *Knödel digraph* $\vec{W}_{\Delta,n}$ be a Knödel graph where each undirected edge is replaced by a symmetric pair of directed edges (an example is shown in Figure 11). In that case, it is easy to see that $\vec{W}_{p-1,n}$ is a broadcast digraph of size $n \times (p - 1)$ for any even n not a power of 2. Hence, in the case $n = 2^p - 2$, the Knödel digraph $\vec{W}_{p-1,n}$ is a MBD.

Theorem 9 *$\vec{W}_{p-1,n}$ and $C'_n(1,3,\ldots 2^{p-1}-1)$ are two non isomorphic classes of MBDs of order $n = 2^p - 2$ for $p \geq 3$.*

Suppose $n = 2^p - 2$, and let us look at the number of neighbours of any vertex u in each graph. By definition, in $\vec{W}_{p-1,n}$, a vertex u has $(p-1)$ neighbours v_i, with, for each of them, a directed edge uv_i and a directed edge v_iu. By definition, in $C'_n(1,3,\ldots,2^{p-1}-1)$, each vertex has at least $(p-1)$ neighbours. Moreover, if every vertex had exactly $(p-1)$ neighbours, then, w.l.o.g., we can focus on vertex v_0. In that case v_{2^p-3} is such that there is a directed edge $v_{2^p-3}v_0$ and a directed edge $v_0v_{2^p-3}$. By definition of $C'_n(1,3,\ldots 2^{p-1}-1)$, the only case where it would be possible is when $2^p - 3 = 2^{p-1} - 1$, that is $p = 2$. Hence $\vec{W}_{p-1,n}$ and $C'_n(1,3,\ldots 2^{p-1}-1)$ are two non isomorphic classes of MBDs for $p \geq 3$.

3.4 Bounds for $\vec{B}(n)$

3.4.1 $n = 2^p + 1$

Theorem 10 *For all $n = 2^p + 1$ with $p \geq 3$, $7 \times 2^{p-2} + 1 \leq \vec{B}(n) \leq 9 \times 2^{p-2} - 2$.*

Proof: When $n = 2^p + 1$, there can be vertices of outdegree 1, and in that case such a vertex, say u, can inform at most n vertices within $(p+1)$ time units. Figure 1 shows the minimum broadcast tree rooted in u in the case $n = 17$, which will help to illustrate the general proof.

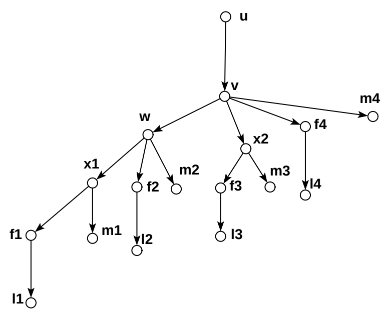

Figure 1: Minimum Broadcast Tree rooted in u of outdegree 1

Let $n = 2^p + 1$ and let u be a vertex of degree 1. Then, as shown in Figure 1, u can inform at most n vertices. In that case, it is not difficult to see that, in the tree, there is 1 vertex v of outdegree p, 1 vertex w of outdegree $(p-1)$, 2 vertices x_1 and x_2 of outdegree $(p-2)$, 4 vertices of outdegree $(p-3)$,..., 2^{p-3} vertices of outdegree 2. Apart from those vertices, there remains $n_1 = 3 \times 2^{p-2}$ vertices in the tree, for which their outdegree is at least 1. Among those n_1 vertices, there are 2^{p-2} leaves m_i such that their father is of outdegree at least 2, and 2^{p-2} leaves l_i such that their father is of outdegree 1. Let us focus on that last class of leaves. Let l be such a leaf, and f its father in the tree. If both are of outdegree 1, the minimum broadcast tree rooted in f would hold strictly less than n vertices. Hence $d^+(f)+d^+(l) \geq 3$. Now if we compute the sum S of all the vertices outdegrees, we get $S \geq 1+p+(p-1)+2(p-2)+4(p-3)+\ldots+2^{p-3}\times 2+2^{p-2}+3\times 2^{p-2}$, that is $S \geq 2 + (2p-1) \times 2^{p-1} - \sum_{i=1}^{p-2}(i \times 2^i)$. As $\vec{B}(n) \geq S$, we get $\vec{B}(n) \geq 7 \times 2^{p-2}$.

Now suppose $\vec{B}(n) = 7 \times 2^{p-2}$. Then the only configuration is $d^+(l) = 1$ for each leaf l of the tree, and $d^+(f) = 2$ for each vertex f such that it was of outdegree 1 in the tree and father of a leaf. Let v be the neighbour of u. As v is the only vertex of outdegree p, it must be neighbour of all the leaves l. Then each directed edge lx will be an edge lv. Now, there remains to add one directed edge fx for every f. Necessarily, at least one of these edges must be f_iu, otherwise no vertex could inform u. Let u be neighbour of f_s. In that case, the minimum broadast tree rooted in f_s holds strictly less than n vertices. Hence $\vec{B}(n) \geq 1 + 7 \times 2^{p-2}$.

The upper bound derives from the following construction : let s be the vertex of outdegree 1, and t be the vertex of outdegree p in the minimum broadcast tree. Let t_1 be the son of t such that $d^+(t_1) = p - 1$, and let l_i be the leaves of the tree such that their father is of outdegree at least 2. To the minimum broadcast tree rooted in s we add all the directed edges $v_i t$ for every vertex $v_i \notin \{s, t, t_1\}$, and all the directed edges $l_i s$ for all i. An example of this construction is given in Figure 2 where $n = 17$. We get the following lemma.

Lemma 1 *The digraph constructed as above is a broadcast digraph and holds* $9 \times 2^{p-2} - 2$ *edges.*

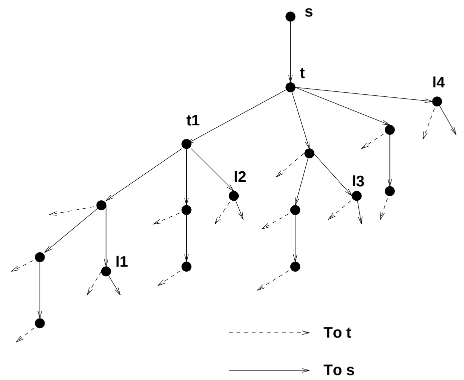

Figure 2: A broadcast digraph on 17 vertices

Proof : The minimum broadcast tree has $(n-1)$ edges. We add $(n-3)$ edges of the form $v_i t$ and 2^{p-2} edges of the form $l_i s$. Hence the number of edges is $2^p + 2^p - 2 + 2^{p-2}$, that is $9 \times 2^{p-2} - 2$.

Let us now prove that this construction gives broadcast digraphs. Let T be the minimum broadcast tree rooted in s which is clearly visible in Figure 2. First, it is easy to see that for vertices s, t and t_1, broadcast can be made in minimum time to all the vertices of the graph. For all the leaves l_i, it is not difficult to see that broadcast can be made in minimum time too : let l_i inform t during the first time unit ; t will then broadcast the information to the rest of the vertices, except s and l_i, the same way as in T. Then s can be informed by l_i during time unit 2, for instance.

It remains to prove that every other vertex v_i can broadcast in this digraph in minimum time. Let us distinguish two classes of vertices. First, consider the vertices v_i such that they are of outdegree at least 2 in T. Hence, the subtree of T rooted in v_i, say T_{v_i}, holds at least one leaf l_j. Let v_i inform t at time unit 1 : t will then broadcast v_i's information to $T - \{T_{v_i} \cup s\}$ as it did in T. Now v_i still needs to inform $\{T_{v_i} \cup s\}$. Recall that in T, v_i could not inform the vertices of T_{v_i} before time unit 3. If v_i informs now the vertices of T_{v_i} from time unit 2, this means that l_j will be informed before the last time unit. Then l_j can inform s during the last time unit, $p+1$, hence v_i has broadcast its information to all the vertices of the digraph.

Now let us consider the vertices w_i of outdegree less or equal to one in T, and let us distinguish two cases : either they are of outdegree 1 in T (let us call those vertices $w1$), or they are of outdegree 0 in T (let us call them $w0$). Figure 3 shows the subtree of T rooted in v_k, father of a $w1$ in T. Note that the other son of v_k is a leaf l_j, as v_k is of outdegree 2 in T.

Let us distinguish the two classes of vertices w_0 and w_1 :

- Let $w0$ inform t at time unit 1. Then t can inform $T - \{w0, s\}$ as it did in T. However, if we swap time units p and $p+1$ during which v_k communicated with, respectively, $w1$ and l_j in T, and if l_j informs s during time unit $p+1$, then $w0$ has informed all the vertices of the digraph in minimum time. We refer to Figure 4 for a better understanding of the method.

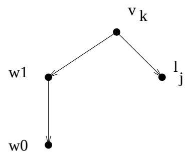

Figure 3: Subtree of T rooted in v_k of outdegree 2

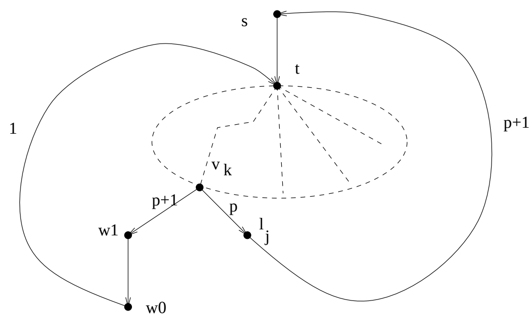

Figure 4: A broadcast scheme for a vertex $w0$

- Analogously, let $w1$ inform t during time unit 1 and let T inform $T - \{w0, w1, s\}$ as it did in T, except for l_j which will be informed at time unit p instead of $p+1$. Then l_j can inform s at time unit $p+1$, and $w1$ can inform $w0$ at time unit, say, 2. Figure 5 shows this broadcast scheme. Hence $w1$ can broadcast its information to all the vertices in minimum time.

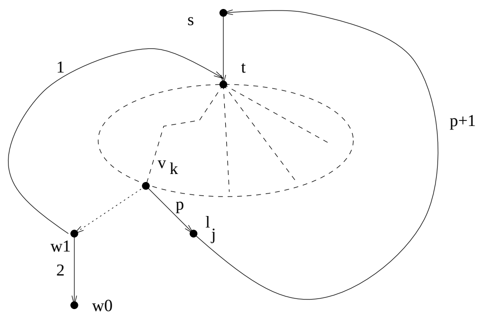

Figure 5: A broadcast scheme for a vertex $w1$

Every vertex of the digraph can broadcast its information to all the vertices in minimum time. Hence, the general construction always give broadcast digraphs, and $\vec{B}(n) \leq 9 \times 2^{p-2} - 2$.

Note that the general upper bound given in this theorem matches the upper bounds given in [LP92] for $n = 9$ and $n = 17$.

3.4.2 $2^p - 2^{p-d} + 2 \leq n \leq 2^p - 5$

Theorem 11 *For all* $2^p - 2^{p-d} + 2 \leq n \leq 2^p$ *with* $1 \leq d \leq p-1$, $\vec{B}(n) \geq (d+1) \times n$.

Proof : If a vertex u is of outdegree d, then it can inform at most $2^p - 2^{p-d} + 1$ vertices within p times units. Hence the result.

3.4.3 $n = 3 \times 2^{p-2} + 1$

Theorem 12 *For all* $n = 3 \times 2^{p-2} + 1$ *with* $p \geq 5$, $\vec{B}(n) \geq 63 \times 2^{p-5}$.

Proof : When $n = 3 \times 2^{p-2} + 1$, it is not difficult to see that a vertex of outdegree 1 cannot inform all the other vertices within p time units. A vertex of outdegree 2, however, can inform all the other vertices within p time units. In that case, the minimum broadcast tree T rooted in such a vertex, say u, holds exaclty n vertices. Figure 6 shows the minimum broadcast tree T rooted in u in the case $n = 25$, which will help to illustrate the general proof.

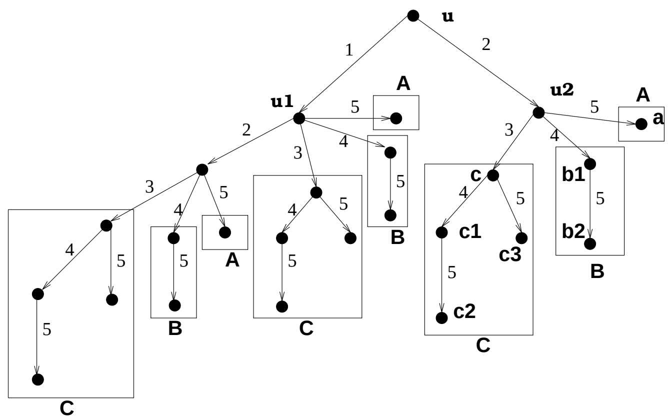

Figure 6: Minimum Broadcast Tree rooted in u of outdegree 2

In that case, the two neighbours of u, say u_1 and u_2, are respectively of outdegree $(p-1)$ and $(p-2)$. And, more generally, if a vertex v is of outdegree 2 with neighbours v_1 and v_2, we have $d^+(v) + d^+(v_i) \geq 5$ for any $i \in \{1, 2\}$.

Now let us consider the minimum broadcast tree rooted in u, T, and let us consider three subtrees of T, A,B,C defined as follows :

- Consider a leaf a such that its father is of outdegree at least 3 in T. $A = \{a\}$;
- Consider a leaf b_2 such that its father b_1 is of outdegree 1 in T. $B = (V_B, A_B)$, where $V_B = \{b_1, b_2\}$ and $A_B = \{b_1 b_2\}$;
- Consider a leaf c_3 such that its father c is of outdegree 2 in T. Let c_1 be the other son of c in T, and c_2 the son of c_1. Let $C = (V_C, A_C)$ where $V_C = \{c, c_1, c_2, c_3\}$ and $A_C = \{cc_1, cc_3, c_1c_2\}$.

Let T_k be the set of subtrees k in T for $k \in \{A, B, C\}$. It is not difficult to see that $|T_A| = |T_B| = |T_C| = 3 \times 2^{p-5}$. Now let us compute S_k, the sum of all the vertices outdegrees of subtree k for $k \in \{A, B, C\}$.

- As stated above, every vertex is of outdegree at least 2. Hence $S_A = d^+(a) \geq 2$.
- It is not difficult to see that $S_B \geq 5$. Indeed, if $d^+(b_1) \geq 3$, as every vertex is of outdegree at least 2, $S_B \geq 5$. If $d^+(b_1) = 2$, we know that $d^+(b_2) \geq 3$. Hence the result.
- We want to show that $S_C \geq 10$. Suppose first $d^+(c) = 2$. Then $d^+(c_1) + d^+(c_3) \geq 7$. Hence $S_C \geq 11$. Now, if $d^+(c) \geq 3$, then we can consider c_1 and c_2 as playing the same role as b_1 and b_2 in B. Hence $d^+(c_1) + d^+(c_2) \geq 5$, that is $S_C \geq 10$.

If we now sum all the outdegrees over all the vertices, we get : $\vec{B}(n) \geq (n-1) + 3 \times 2^{p-5}(2 + (5 - 1) + (10 - 3))$, that is $\vec{B}(n) \geq 63 \times 2^{p-5}$.

3.4.4 $n = 2^p - 3$

Theorem 13 *For all* $n = 2^p - 3$ *with* $p \geq 4$, $n \times (p-2) + 3 \leq \vec{B}(n) \leq n \times (p-1) - 1$.

Proof : In [LP92], Liestman and Peters gave an equivalent of Farley's two-way split method for broadcast digraphs. This method gives the following formula : $\vec{B}(n) \leq \vec{B}(n_1) + \vec{B}(n_2) + 2n_2$, where $n_1 + n_2 = n \geq 4$, $n_1 \geq n_2$ and $\lceil log_2 n_1 \rceil = \lceil log_2 n_2 \rceil = \lceil log_2 n \rceil - 1$. Using this method, we get the upper bound on $\vec{B}(2^p - 3)$ where $n_1 = 2^{p-1} - 1$ and $n_2 = 2^{p-1} - 2$.

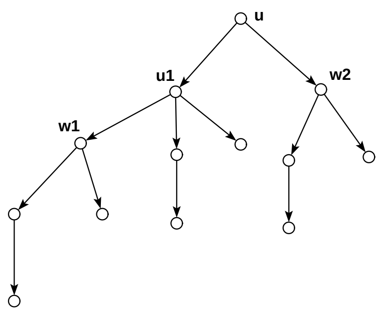

Figure 7: Minimum Broadcast Tree on 13 vertices

If we have a vertex u of outdegree $(p-2)$, it will be able to inform exactly $n = 2^p - 3$ vertices within p time units, as shown in Figure 7 for the case $n = 13$. But this implies that the vertex informed by u after the first time unit, say u_1, is of outdegree $(p-1)$ at least. In the broadcast tree rooted in u, there are two other vertices w_1 and w_2 which are of outdegree at least $(p-2)$. W.l.o.g., let us consider w_1 : either w_1 is of outdegree at least $(p-1)$, or it is of outdegree $(p-2)$ and one of its sons in the tree is of outdegree at least $(p-1)$. In every case, we show that at least three vertices in the graph are of outdegree at least $(p-1)$, hence the result.

3.4.5 $n = 2^p - 4$

Theorem 14 *For all* $n = 2^p - 4$ *with* $p \geq 4$, $n \times (p-2) \leq \vec{B}(n) \leq n \times (p - \frac{3}{2})$.

Proof : Any vertex of outdegree strictly less than $(p-2)$ can inform up to $2^p - 7$ vertices, hence the lower bound. The upper bound derives from an upper bound given in [Sac96] in the undirected case. Indeed, Saclé has shown that $B(2^p - 4) \leq \frac{n}{2} \times (p - \frac{3}{2})$. As $\vec{B}(n) \leq 2 \times B(n)$ for any n, we get the result.

Remark : It would be possible to go on for $n = 2^p - 5$, $n = 2^p - 6$, etc. However, for $n = 2^p - 3$ and $n = 2^p - 4$, the bounds presented above give new results in the range 1..32 (namely, $n = 28$ and $n = 29$), while this is not the case for $n \leq 2^p - 5$.

4 Particular cases

This section is devoted to the values of $\vec{B}(n)$ for $1 \leq n \leq 32$, which Liestman and Peters have studied in [LP92]. A few improvements and/or addings are presented below.

4.1 New Minimum Broadcast Digraphs

It is interesting to see that the constructions given above in this article provide MBDs which are not necessarily isomorphic to the ones provided in [LP92]. We are going to detail such graphs of order n for n in the range 1..32.

Theorem 15 $C'_6(1,3)$ *is a MBD of order 6 non isomorphic to the one given in [LP92].*

Proof : In [LP92], the MBD on 6 vertices used to prove optimality is based on the undirected cycle where each undirected edge has been replaced by a pair of symmetric directed edges. Note that this MBD can also be seen as the circulant digraph $C'_6(1,5)$. The MBD provided in Theorem 8 for $n = 2^p - 2$ where $p = 3$ is $C'_6(1,3)$, shown in Figure 8. It is easy to see that $C'_6(1,5)$ is not isomorphic to $C'_6(1,3)$, because every vertex in $C'_6(1,3)$ has three neighbours, while every vertex in the MBD displayed in [LP92] has two neighbours.

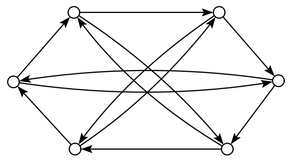

Figure 8: A MBD on 6 vertices

Remark : The MBD on 7 vertices shown in [LP92] is $C'_7(1,3)$.

Theorem 16 *The graph shown in Figure 9 is a MBD of order 9 non isomorphic to the one given in [LP92].*

Proof : Liestman and Peters [LP92] proved that $\vec{B}(9) = 16$ and gave one MBD on 9 vertices. The construction provided in proof of Theorem 10 gives broadcast digraphs with $2^p + 1$ vertices and $9 \times 2^{p-2} - 2$ edges. Hence, in the case $p = 3$, this construction gives a MBD on 9 vertices. Moreover, it is not isomorphic to the MBD presented in [LP92], as in our case, vertex t is of indegree 7 while no vertex is of indegree more than 6 in the MBD presented in [LP92].

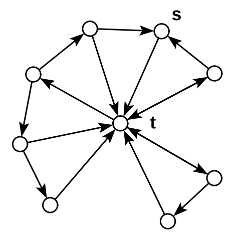

Figure 9: A MBD on 9 vertices

Theorem 17 *In the case* $n = 14$ *:*

- *The circulant digraph* $C'_{14}(1,3,7)$ *shown in Figure 10 is a MBD of order 14 non isomorphic to the one given in [LP92].*
- *Similarly, the Knödel digraph* $\vec{W}_{3,14}$ *shown in Figure 11 is a MBD on 14 vertices non isomorphic to the one given in [LP92].*

Remark : As seen in Section 3.3, we know that $C'_{14}(1,3,7)$ and $\vec{W}_{3,14}$ are non isomorphic.

Proof : In [LP92], the MBD on 14 vertices used to prove optimality can be seen as $C'_{14}(1,5,11)$, as ours is $C'_{14}(1,3,7)$, as shown in Figure 10. To show that $C'_{14}(1,5,11)$ is not isomorphic to $C'_{14}(1,3,7)$, let us count the number of neighbours of each vertex in each graph. Let us consider

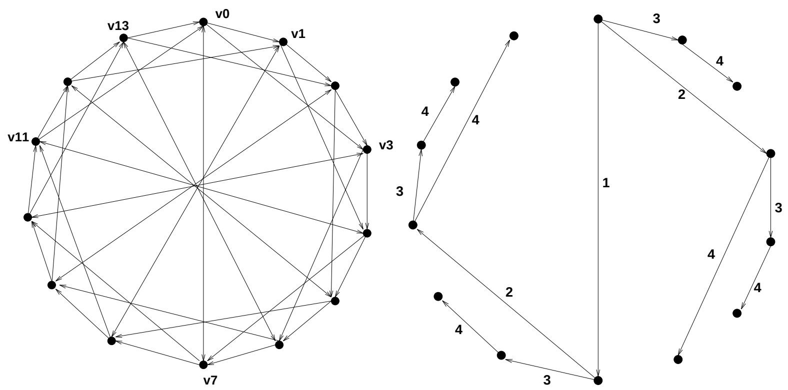

Figure 10: (a) $C'_{14}(1,3,7)$: a MBD on 14 vertices and (b) a broadcast scheme

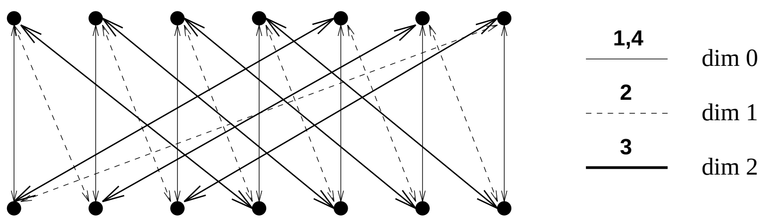

Figure 11: $\vec{W}_{3,14}$ and (right) a broadcast scheme

$C'_{14}(1,5,11)$ and, say, vertex v_0 ; it has 6 neighbours, namely v_1, v_5, v_{11}, v_{13}, v_9 and v_3. Analogously, we can count the number of neighbours of vertex v_0 in $C'_{14}(1,3,7)$. It is easy to see that there are only 5 dinstinct neighbours : v_1, v_3, v_7, v_{13} and v_{11}, since vertex v_7 is involved twice in the neighbourhood of vertex v_0. Hence $C'_{14}(1,5,11)$ is not isomorphic to $C'_{14}(1,3,7)$.

Similarly, we can see that any vertex of $\vec{W}_{3,14}$ has only three neighbours, hence it is not isomorphic to $C'_{14}(1,5,11)$.

Remark : It would be interesting to see if the MBDs $C'_n(1,3,\ldots,2^{\lfloor log_2(n)\rfloor}-1)$ and the MBDs given in [LP92] are or are not isomorphic for $n=15$, $n=30$ since :

- For $n=15$, the MBD provided in [LP92] is $C'_{15}(1,7,12)$.
- For $n=30$, the MBD provided in [LP92] is the result of Farley's two-way split method on two MBDs of order 15.

However, we can give a new MBD_{30} thanks to the following theorem.

Theorem 18 *The Knödel digraph $\vec{W}_{4,30}$ is a MBD on 30 vertices non isomorphic to the one given in [LP92].*

Proof : In the case $n=30$, the MBD provided in [LP92] is the result of Farley's two-way split method on two MBDs of order 15. Moreover, the MBD on 15 vertices given in [LP92] is $C'_{15}(1,7,12)$. It is not difficult to see that, in that case, any vertex v of the MBD_{30} given in [LP92] has 7 neighbours, while any vertex v in $\vec{W}_{4,30}$ has 4 neighbours. Hence the result.

4.2 New bounds for $\vec{B}(n)$ in the range 1..32

Theorem 19 $\vec{B}(17) \geq 29$.

Proof : This is a direct consequence of Theorem 10, where $n = 2^p + 1$ with $p = 4$.

Theorem 20 $\vec{B}(25) \geq 63$.

Proof : This comes from the application of Theorem 12 for $p = 5$.

Theorem 21 $\vec{B}(27) \leq 88$.

Proof : Saclé [Sac96] showed that $B(27) = 44$ in the undirected case. As $\vec{B}(n) \leq 2 \times B(n)$ for any n, we get the result.

Theorem 22 $\vec{B}(28) \leq 96$.

Proof : This is a direct consequence of Theorem 14, where $n = 2^p - 4$ with $p = 5$.

Theorem 23 $90 \leq \vec{B}(29) \leq 104$.

Proof : The lower bound is a direct consequence of Theorem 13, where $n = 2^p - 3$ and $p = 5$. The upper bound comes from [Sac96], where it has been shown that $B(29) = 52$. As $\vec{B}(n) \leq 2 \times B(n)$ holds for any n, we get the result.

Theorem 24 $\vec{B}(31) = 124$.

Proof : This is a straightforward application of Theorem 8, where $n = 2^p - 1$ with $p = 5$. Figure 12 shows a possible broadcast sheme in the circulant digraph $C'_n(1, 3, 7, 15)$.

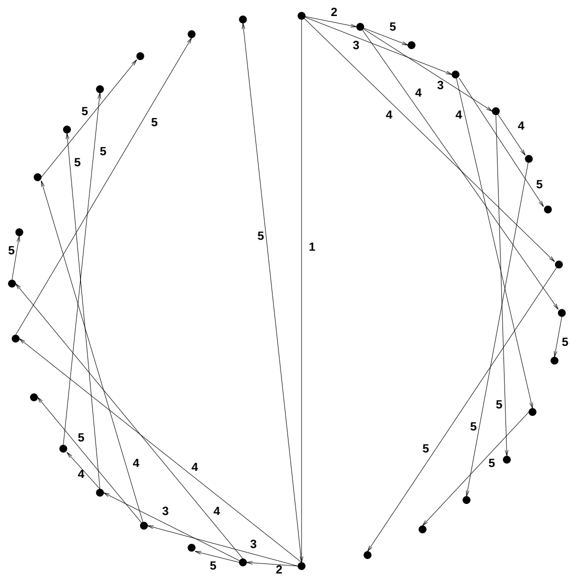

Figure 12: A broadcast scheme for $C'_{31}(1, 3, 7, 15)$

The table displayed on Figure 13 shows respectively lower and upper bounds for $\vec{B}(n)$ for n in the range 1..32. The asterisk indicates optimality, and bounds printed in bold characters indicate new results.

n	Lower	Upper	n	Lower	Upper	n	Lower	Upper	n	Lower	Upper
1	0	0*	9	16	16*	17	**29**	34	25	**63**	75
2	2	2*	10	20	20*	18	36	36*	26	78	78*
3	3	3*	11	22	22*	19	38	39	27	81	**88**
4	8	8*	12	24	24*	20	40	40*	28	84	**96**
5	7	7*	13	29	33	21	43	53	29	**90**	**104**
6	12	12*	14	42	42*	22	45	55	30	120	120*
7	14	14*	15	45	45*	23	47	64	31	**124**	**124***
8	24	24*	16	64	64*	24	49	66	32	160	160*

Figure 13: Sum-up of known results for $1 \leq n \leq 32$

5 Conclusion

Thanks to the construction provided by Park and Chwa, [PC94] it has been possible to determine $\vec{B}(n)$ for two classes of infinite values of n, namely $n = 2^p - 1$ and $n = 2^p - 2$. This has been made possible because any vertex of outdegree strictly less than $(p-1)$ can inform at most $2^p - 3$ vertices, and because a $(p-1)$-regular digraph can inform up to $2^p - 1$ vertices. It would be interesting to go further in this study, noticing that any vertex of outdegree strictly less than $(p-2)$ can inform at most $2^p - 7$ vertices, and that a $(p-2)$-regular digraph could inform up to $2^p - 4$ vertices. Hence, if we manage to find a class of $(p-2)$-regular digraphs that are broadcast digraphs for any $2^p - 6 \leq n \leq 2^p - 4$, we would get the exact values of $\vec{B}(n)$. Note that it is true for some small values of n, namely $n = 10$, $n = 11$, $n = 12$ and $n = 26$. Analogously, with a $(p-3)$-regular broadcast digraph, we could determine $\vec{B}(n)$ for $2^p - 14 \leq n \leq 2^p - 12$, as it is the case for $n = 18$ and $n = 20$. It is interesting to notice that this theory could not go further, since a $(p-4)$-regular digraph could inform at most $2^p - 32$ vertices, while a vertex of outdegree $(p-5)$ can inform up to $2^p - 29$ vertices.

Moreover, note that many bounds that were given in [LP92] are reached by the more general formulas presented in Section 3, while some others have been improved.

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