$L(p,q)$ Labeling of d-Dimensional Grids*

Guillaume Fertin[1], André Raspaud[2]

[1]LINA FRE CNRS 2729, Université de Nantes
2 rue de la Houssinière - BP 92208 - F44322 Nantes Cedex 3
[2]LaBRI UMR 5800, Université Bordeaux 1
351 Cours de la Libération - F33405 Talence Cedex

fertin@lina.univ-nantes.fr, raspaud@labri.fr

Abstract

In this paper, we address the problem of λ labelings, that was introduced in the context of frequency assignment for telecommunication networks. In this model, stations within a given radius r must use frequencies that differ at least by a value p, while stations that are within a larger radius $r' > r$ must use frequencies that differ by at least another value q. The aim is to minimize the span of frequencies used in the network. This can be modeled by a graph coloring problem, called the $L(p,q)$ labeling, where one wants to label vertices of the graph G modeling the network by integers in the range $[0; M]$, in such a way that (1) neighbors in G are assigned colors differing by at least p and (2) vertices at distance 2 in G are assigned colors differing by at least q, while minimizing the value of M. M is then called the λ number of G, and is denoted by $\lambda_q^p(G)$.

In this paper, we study the $L(p,q)$ labeling for a specific class of networks, namely the d-dimensional grid $G_d = G[n_1, n_2 \dots n_d]$. We give bounds on the value of the λ number of an $L(p,q)$ labeling for any $d \geq 1$ and $p,q \geq 0$. Some of these results are optimal (namely, in the following cases : (1) $p = 0$, (2) $q = 0$, (3) $q = 1$ (4) $p,q \geq 1$, $p = \alpha \cdot q$ with $1 \leq \alpha \leq 2d$ and (5) $p \geq 2dq + 1$) ; when the results we obtain are not optimal, we observe that the bounds differ by an additive factor never exceeding $2q - 2$. The optimal result we obtain in the case $q = 1$ answers an open problem stated by Dubhashi et al. [DMP+02], and generalizes results from [BPT00] and [DMP+02]. We also apply our results to get upper bounds for the $L(p,q)$ labeling of d-dimensional hypercubes.

1 Introduction

In this paper, we study the *frequency assignment problem*, originally introduced in [Hal80], where radio transmitters that are geographically close may interfere if they are assigned close frequencies. This problem arises in mobile or wireless networks. Generally, this problem is modeled by a graph coloring problem, where the transmitters are the vertices, and an edge joins two transmitters that are sufficiently close to potentially interfere. The aim here is to color (i.e. give an integer value, corresponding to the frequency) the vertices of the graph in such a way that :

- any two neighbors (transmitters that are very close) are assigned colors (frequencies) that differ by a parameter at least p ;
- any two vertices at distance 2 (transmitters that are close) are assigned colors (frequencies) that differ by a parameter at least q ;

*Part of this work was done while both authors were visiting the Universitat Politècnica de Catalunya in Barcelona. This visit was supported by the Egide Spanish-French exchange program PICASSO Number 04295SG.

- the greatest value for the colors is minimized.

It has been proved that under this model, we could assume the colors to be integers, starting from 0 [GY92]. In that case, the minimum range of frequencies that is necessary to assign the vertices of a graph G is denoted $\lambda_q^p(G)$, and the problem itself is usually called the $L(p, q)$ labeling problem. The frequency assignment problem has been studied in many different specific topologies [GY92, Sak94, WGM95, BPT00, BKTvL00, CKK$^+$02, MS02, BPT02, Kra, FKP01]. The case $p = 2$ and $q = 1$ is the most widely studied (see for instance [CK96, JNS$^+$00, Jha00, CP01, FKK01]). Some variants of the model also exist, such as the following generalization, called the $L(\delta_1, \dots \delta_k)$ labeling problem, where one gives k constraints on the k first distances (any two vertices at distance $1 \le i \le k$ in G must be assigned colors differing by at least δ_i). One of the issues also considered in the frequency assignment problem is the *no-hole property*, where one wants to know whether a given coloring with span M uses all the possible colors in the range $[0; M]$.

In this paper, we focus on the $L(p, q)$ labeling problem, and we study in Section 2 the case of the $L(p, q)$ labeling in the d-dimensional grid G_d. We first address in Section 2.1 the cases where $p = 0$ or $q = 0$. In Section 2.2, we give results for the $L(p, q)$ labeling of G_d for any $p, q, d \ge 1$. We give lower and upper bounds on $\lambda_q^p(G_d)$, and show that in some cases, these bounds coincide. Notably, in the case $q = 1$, the results we obtain are optimal ; this answers an open problem stated by Dubhashi et al. in [DMP$^+$02], and generalizes results from [BPT00] and [DMP$^+$02]. The results we give are also optimal when $p = 0$, $q = 0$, $p = \alpha \cdot q$ with $1 \le \alpha \le 2d$, and $p \ge 2dq + 1$. We prove that in some cases (namely, when $1 \le p \le 2dq$), one of the colorings we propose satisfies the no-hole property. We also apply our results to get upper bounds for the $L(p, q)$ labeling of d-dimensional hypercubes.

2 $L(p, q)$ labeling of G_d

We now turn to the case of the λ labeling problem with two constraints on the distances, in a particular network topology, namely the d-dimensional grid $G_d = G[n_1, n_2 \dots n_d]$. We first recall the definition of such a network.

Definition 1 *Let $d \in \mathbb{N}$ and $(n_1, \dots, n_d) \in \mathbb{N}^d$, with $n_i \ge 2$ for any $1 \le i \le d$. The $d-$*dimensional *grid of lengths $n_1, \dots, n_d$, denoted by $G_d(n_1, \dots, n_d)$, is the following graph:*

$$\begin{array}{rcl} V(G_d(n_1, \dots, n_d)) & = & [1, n_1] \times [1, n_2] \times \dots \times [1, n_d] \\ E(G_d(n_1, \dots, n_d)) & = & \{\{u, v\} \mid u = (u_1, \dots, u_d), v = (v_1, \dots, v_d), \textit{ and there exists } i_0 \textit{ such that} \\ & & \forall i \ne i_0, u_i = v_i, \textit{ and } |u_{i_0} - v_{i_0}| = 1\} \end{array}$$

We first address the $L(p, q)$ labeling of G_d in the special cases where $p = 0$ (resp. $q = 0$) in Section 2.1. We then address the more general case where $p, q \ge 1$ in Section 2.2.

2.1 $L(p, q)$ labeling when $p = 0$ or $q = 0$

Proposition 1 *For any $p \ge 0$ and $d \ge 1$, $\lambda_0^p(G_d) = p$.*

Proof : Consider an optimal $L(p, 0)$ labeling of the vertices of G_d. Clearly, there must exist a vertex, say v, with color $c(v) = 0$ (if not, then $\lambda_0^p(G_d)$ could be reduced by at least 1). Then, any neighbor of v must have a color greater than or equal to p. Thus, $\lambda_0^p(G_d) \ge p$.

Now, since G_d is bipartite, we have that $\lambda_0^p(G_d) \le p$. Indeed, let us define the following coloring: for any vertex $v = (x_1, x_2 \dots x_d)$, if $\sum_{i=1}^d x_i \equiv 0 \mod 2$, then $c(v) = 0$, otherwise $c(v) = p$. Since $q = 0$, it suffices to check that any two neighbors are assigned colors that differ by at least p. It is clearly the case here, since any two neighbors in G_d have the sum of their coordinates of different parity, and thus will be assigned different colors. Since the colors are taken in the set $\{0, p\}$, we have that any two neighbors u and v in G_d satisfy $|c(u) - c(v)| \ge p$. Hence we have $\lambda_0^p(G_d) = p$.
□

Proposition 2 *For any $q \geq 0$ and $d \geq 1$, $\lambda_q^0(G_d) = (2d-1)q$.*

Proof : Consider a vertex v of degree $2d$ in G_d, and let $c(v)$ be its color in an optimal $L(0,q)$ labeling of G_d. In that case, at most one neighbor of v, say w, can satisfy $c(v) = c(w)$. Thus, there remains $2d-1$ neighbors of u to color, and since those vertices are at distance 2 from w and from each other, they must use colors that are pairwise q away. Thus, if we assume w.l.o.g. $c(v) = c(w) = 0$, then the best we can expect for those $2d-1$ vertices is that they be assigned colors in the set $q, 2q \dots (2d-1)q$. Hence, $\lambda_q^0(G_d) \geq (2d-1)q$.

Now we give an $L(0,q)$ labeling of G_d that uses colors in the set $\{0, p, 2q \dots (2d-1)q\}$: for any vertex $v = (x_1, x_2 \dots x_d)$ of G_d, we define

$$c(v) = (dq\lfloor \frac{x_d}{2} \rfloor + \sum_{i=1}^{d-1} iqx_i) \ mod \ 2dq$$

Since $p = 0$, we only need to consider two vertices u and v lying at distance 2 in G_d, thus differing on two coordinates, say x_i and x_j, $1 \leq i \leq j \leq d$. W.l.o.g. we can consider only two cases, supposing $u = (x_1, \dots x_i \dots x_j \dots x_d)$: (1) $v = (x_1, \dots x_i + 1 \dots x_j + 1 \dots x_d)$ (where possibly $i = j$) and (2) $v = (x_1, \dots x_i + 1 \dots x_j - 1 \dots x_d)$ (where $i \neq j$). In case (1), we will distinguish two cases : (1a) $i = j$ and (1b) $i \neq j$. In case (1a), we have again two cases two consider : first, if $i = j = d$, then clearly $|c(v) - c(u)| = dq$, and the condition is satisfied since $d \geq 1$. If $i = j \neq d$, then $|c(v) - c(u)| = 2iq$, since $1 \leq i \leq d-1$. Thus the condition is satisfied as well. Now we turn to case (1b), where $i \neq j$. W.l.o.g., we thus consider $1 \leq i < j \leq d$. If $j = d$, then depending on the parity of x_d, we either obtain $|c(v) - c(u)| = iq$ (for even x_d) or $|c(v) - c(u)| = (i+d)q$ (for odd x_d). In both cases, since $i \leq d-1$, we see that the condition is satisfied. Now, if $j \neq d$, then $|c(v) - c(u)| = (i+j)q$, which also satisfies the condition that the colors differ by at least q.

In case (2), we know that necessarily $i \neq j$, and thus we consider as above $1 \leq i < j \leq d$. First, if $j = d$, then depending on the parity of x_d, we either obtain $|c(v) - c(u)| = iq$ (for even x_d) or $|c(v) - c(u)| = (d-i)q$ (for odd x_d). In both cases, the condition is satisfied since $1 \leq i \leq d-1$. Now, if $j \neq d$, we get $|c(v) - c(u)| = (j-i)q$, which also satisfies the conditions since $1 \leq i < j \leq d$.

Overall, we see that this coloring satisfies the distance 2 condition, and thus is valid to $L(0,q)$ label G_d. Since it uses colors in the set $\{0, q, 2q, \dots (2d-1)q\}$, we conclude that $\lambda_q^0(G_d) \leq (2d-1)q$. Hence, altogether we have $\lambda_q^0(G_d) = (2d-1)q$. □

We note that except in specific cases, the colorings we have given above do not satisfy the no-hole property (we recall that the no-hole property holds when all colors in the range $[0; \lambda_q^p]$ are used). Indeed, in the case $q = 0$, only colors 0 and p are used, thus the coloring is not no-hole for any $p \geq 2$. Similarly, in the case $p = 0$, the colors used are taken in the set $\{0, q, 2q \dots (2d-1)q\}$, thus the coloring is not no-hole for any $q \geq 2$.

2.2 $L(p,q)$ labeling when $p, q \geq 1$

We now address the $L(p,q)$ labeling of G_d, for any values of $p, q \geq 1$ and $d \geq 1$. First, we note that we can obtain two trivial upper bounds on $\lambda_q^p(G_d)$; this is described in the two following observations.

Observation 1 *For any $p, q, d \geq 1$, $\lambda_q^p(G_d) \leq q \cdot \lambda_1^{\lceil \frac{p}{q} \rceil}(G_d)$.*

Proof : The key idea here is to first achieve an $L(\alpha, 1)$ coloring $\mathcal{C}$ of G_d (thus with λ number $\lambda_1^\alpha(G_d)$), and to get another coloring $\mathcal{C}'$ by multiplying the color of each vertex by q. In that case, two vertices at distance 2 in G_d, which were assigned different colors by $\mathcal{C}$, are assigned colors differing by at least q in $\mathcal{C}'$. Moreover, two neighbors in G_d are now assigned colors differing by at least αq. Since we want $\alpha q \geq p$, it suffices to take $\alpha = \lceil \frac{p}{q} \rceil$. □

There exists another upper bound for $\lambda_q^p(G_d)$, that relies on the $L(1,1)$ labeling of G_d. We refer for this to Observation 2 below.

Observation 2 *For any $p, q, d \geq 1$, $\lambda_q^p(G_d) \leq \max\{p, q\} \cdot 2d$.*

Proof : Let $\mathcal{C}$ denote the coloring derived from an optimal $L(1,1)$ of G_d, and let $\mathcal{C}'$ be a new coloring obtained from $\mathcal{C}$ by multiplying every color by $\max\{p, q\}$. In that case, any two vertices at distance 1 (resp. 2) in G_d are assigned colors that are at least p (resp. at least q) apart. Hence this is an $L(p, q)$ labeling of G_d. Since we know (cf. for instance [FGR03] or Theorem 1 below) that $\lambda_1^1(G_d) = 2d$, we conclude that $\lambda_q^p(G_d) \leq \max\{p, q\} \cdot 2d$. □

The two above mentioned simple observations present the disadvantage to be based upon an existing labeling (an $L(p, 1)$ labeling for Observation 1, and an $L(1, 1)$ labeling for Observation 2). In the following, we study the problem in more details, and define upper and lower bounds on $\lambda_q^p(G_d)$ for all values of $p, q, d \geq 1$ (resp. in Lemmas 1 and 2). These results directly imply Theorem 1.

Lemma 1 *For any $p, q, d \geq 1$,*

- $\lambda_q^p(G_d) \geq 2p + (2d-2)q$ *when* $1 \leq p \leq 2dq$
- $\lambda_q^p(G_d) \geq p + (4d-2)q$ *when* $p \geq 2dq + 1$

Proof : Suppose that it is possible to $L(p, q)$ label the vertices of G_d with M colors, with $M \leq 2p + (2d-2)q - 1$. We will first show that in that case, no vertex of degree $2d$ in G_d can be assigned a color in the range $[p-1; p + (2d-1)q - 1]$.
Indeed, suppose there exists a vertex $u \in V(G_d)$ such that u is assigned color $p + x$, with $-1 \leq x \leq (2d-1)q - 1$. Then, all its neighbors must be assigned a color in the range $[0; x] \cup [2p + x; M]$, because of the gap of at least p that must exist between neighbors. Within this range, one must be able to get $2d$ values, each pair of which differ of at least q. Let us distinguish two cases : (i) $x = -1$ and (ii) $x \geq 0$. In case (i), it is clear that all the colors must be in the range $[2p-1; M]$. In other words, if we want to be able to assign the $2d$ colors of the neighbors, we must have $2p - 1 + (2d-1)q \leq M$. However, we supposed $M \leq 2p + (2d-2)q - 1$, hence the contradiction since $q \geq 1$. Now suppose that (ii) $x \geq 0$; we distinguish two more cases : (ii-1) $x = kq$ with $k \geq 0$ and (ii-2) $x = kq - i$, with $k \geq 1$ and $1 \leq i \leq q-1$. In case (ii-1), we can use $(k+1)$ colors in the range $[0; kq]$ (more precisely, colors $0, q, 2q \ldots kq$). Hence there remains $2d - (k-1)$ colors to get in the range $[2p + x; M]$. For this, we must have $2p + x + (2d - (k-1) - 1)q \leq M$. This gives $2p + (2d-2)q \leq M$, a contradiction. In case (ii-2), only k colors can be assigned in the range $[0; x]$. Thus $2d - k$ colors must be assigned in the range $[2p + x; M]$, which can be the case only if $2p + x + (2d - k - 1)q \leq M$. This can happen only when $i \geq q + 1$, a contradiction too. Thus we conclude that if $\lambda_q^p(G_d) = M$, no vertex of degree $2d$ in G_d can be assigned a color in the range $[p-1; p + (2d-1)q - 1]$.
In other words, if such a coloring exists, all vertices of degree $2d$ are assigned colors in the range $[0; p-2] \cup [p + (2d-1)q; M]$. Let $I_1 = [0; p-2]$ and $I_2 = [p + (2d-1)q; M]$, with $M = 2p + (2d-2)q - j$, $j \geq 1$. Clearly, I_1 contains $p-1$ integers, and I_2 contains $p - q - j + 1 < p$ integers (since $j, q \geq 1$). This means that if a vertex u of degree $2d$ in G_d is assigned a color in I_1 (resp. I_2), all its neighbors must be assigned colors in I_2 (resp. I_1) – supposing that all the neighbors of u are of degree $2d$, which happens if G_d is "big" enough. However, in order for I_1 (resp. I_2) to support $2d$ colors that, pairwise, admit a gap of q, the two following conditions must be fulfilled :

(1) $(2d-1)q \leq p - 2$ and

(2) $p + (2d-1)q + (2d-1)q \leq M$

In other words, we must have (1') $p \geq (2d-1)q + 2$ and (2') $p \geq 2dq + j$. Since $j, q \geq 1$, condition (2') implies condition (1'). Thus, in order to have a valid $L(p, q)$ labeling with $\lambda_q^p(G_d) = M$, we must have $p \geq 2dq + j$ with $j \geq 1$. However, we supposed $p \leq 2dq$, hence the contradiction.
Overall, we have proved that for a sufficiently large grid, and when $1 \leq p \leq 2dq$, $\lambda_q^p(G_d) \geq 2p + (2d-2)q$, and the first part of the lemma is proved.

Now suppose that $p \geq 2dq + 1$. Suppose that $\lambda_q^p(G_d) = M'$, with $M' < p + (4d - 2)q$. Since we supposed that $p \geq 2dq + 1$, we have $p + (4d - 2)q < 2p + (2d - 2)q$, we can reuse one of the previous arguments and conclude that no vertex of degree $2d$ can be assigned a color in the range $[p - 1; p + (2d - 1)q - 1]$. Hence all the vertices of degree $2d$ must be assigned colors in $[0; p - 2] \cup [p + (2d - 1)q; M']$. We also use one of the previous arguments here to say that in that case we must have (1) $(2d - 1)q \leq p - 2$ and (2) $p + (2d - 1)q \leq M'$. However, (2) is not satisfied, hence the contradiction. We then conclude that necessarily, in the case $p \geq 2dq + 1$, $\lambda_q^p(G_d) \geq p + (4d - 2)q$. □

Lemma 2 *For any* $p, q, d \geq 1$,

- $\lambda_q^p(G_d) \leq 2dq$ *when* $2 \leq 2p < q$
- $\lambda_q^p(G_d) \leq 2p + (2d - 1)q - 1$ *when* $1 \leq q \leq 2p \leq 4dq$
- $\lambda_q^p(G_d) \leq p + (4d - 2)q$ *when* $p \geq 2dq + 1$

Proof : Let $p, q, d \geq 1$. In order to prove these upper bounds, we give an ad hoc coloring in each of the two cases, and show that it respects the constraints at distances 1 and 2.

Case 1: $2 \leq 2p < q$. For any vertex $v = (x_1 \dots x_d)$ in G_d, with $x_i \geq 0$ for any $1 \leq i \leq d$, we assign to v color $c(v)$ defined as follows :

$$c(v) = (\sum_{i=1}^{d} qix_i) \; mod \; (2d + 1)q$$

We are going to prove that this coloring is an $L(p, q)$ labeling of G_d. For this, we distinguish two cases :

- u and v are neighbors in G_d, thus they differ on one coordinate x_i, $1 \leq i \leq d$. W.l.o.g., suppose $u = (x_1, \dots x_i \dots x_d)$ and $v = (x_1, \dots x_i + 1 \dots x_d)$. In that case $|c(v) - c(u)| = iq$ mod $(2d + 1)q$. Since $1 \leq i \leq d$, we have that $|c(v) - c(u)| \geq q$. However, we supposed $p < 2q$, thus we conclude $q > p$ and $|c(v) - c(u)| > p$.
- u and v lie at distance 2 in G_d, thus they differ on two coordinates x_i and x_j, $1 \leq i \leq j \leq d$. W.l.o.g. we can consider only two cases, supposing $u = (x_1, \dots x_i \dots x_j \dots x_d)$: (1) $v = (x_1, \dots x_i{+}1 \dots x_j{+}1 \dots x_d)$ (where possibly $i = j$) and (2) $v = (x_1, \dots x_i{+}1 \dots x_j{-}1 \dots x_d)$ (where $i \neq j$). In case (1) we have $|c(v) - c(u)| = (i + j)q$ mod $(2d + 1)q$. Since $1 \leq i, j \leq d$, we conclude that $|c(v) - c(u)| \geq 2q$, and thus $|c(v) - c(u)| \geq q$. In case (2), we have $|c(v) - c(u)| = (j - i)q$ mod $(2d + 1)q$. Since $1 \leq i < j \leq d$ (because we supposed that $j \geq i$, and since we are in case (2), $i \neq j$), we have that $|c(v) - c(u)| \geq q$. Hence the constraint is satisfied in both cases.

Hence, the above mentioned coloring is an $L(p, q)$ labeling of the grid in the case $p, q, d \geq 1$ and $2p < q$. Since it uses colors in the set $\{0, q, 2q, \dots 2dq\}$, we conclude that $\lambda_q^p(G_d) \leq 2dq$.

Case 2: $1 \leq q \leq 2p \leq 4dq$. For any vertex $v = (x_1 \dots x_d)$ in G_d, with $x_i \geq 0$ for any $1 \leq i \leq d$, we assign to v color $c(v)$ defined as follows :

$$c(v) = (\sum_{i=1}^{d} (p + (i - 1) \cdot q)x_i) \; mod \; (2p + (2d - 1)q)$$

We are going to prove that this coloring is an $L(p, q)$ labeling of G_d. For this, we distinguish two cases :

- u and v are neighbors in G_d, thus they differ on one coordinate x_i, $1 \le i \le d$. W.l.o.g., suppose $u = (x_1, \dots x_i \dots x_d)$ and $v = (x_1, \dots x_i{+}1 \dots x_d)$. Thus $|c(v) - c(u)| = (p + (i-1)q)$. Since $1 \le i \le d$, we have that $|c(v) - c(u)| \ge p$.

- u and v lie at distance 2 in G_d, thus they differ on two coordinates x_i and x_j, $1 \le i \le j \le d$. W.l.o.g. we can consider only two cases, supposing $u = (x_1, \dots x_i \dots x_j \dots x_d)$: (1) $v = (x_1, \dots x_i{+}1 \dots x_j{+}1 \dots x_d)$ (where possibly $i = j$) and (2) $v = (x_1, \dots x_i{+}1 \dots x_j{-}1 \dots x_d)$ (where $i \neq j$). In case (1), $|c(u) - c(v)| = 2p + (i + j - 2)q$. In that case, $|c(u) - c(v)| \ge q$, except maybe when $i = j = 1$. However, when $i = j = 1$, then $|c(u) - c(v)| = 2p$, and by hypothesis we know that $2p \ge q$. Thus $|c(u) - c(v)| \ge q$ in all the cases. In case (2), since $i \neq j$, we have that $j > i$. Then we have $|c(u) - c(v)| = (j - i)q$. Thus, for any two vertices $u, v \in V(G_d)$ that lie at distance 2, we have $|c(u) - c(v)| \ge q$.

Hence, we have proved that the above mentioned coloring is an $L(p, q)$ labeling of the grid in the case $p, q, d \ge 1$, $1 \le q \le 2p \le 4dq$. Since it uses colors in the range $[0; 2p + (2d-1)q - 1]$, we conclude that $\lambda_q^p(G_d) \le 2p + (2d-1)q - 1$.

Case 3: $p \ge 2dq + 1$. In this case, any vertex having an even sum of coordinates will be assigned a color in the range $[p + (2d-1)q; p + (4d-2)q]$, while any vertex having an odd sum of coordinates will be assigned a color in the range $[0; (2d-1)q]$. More precisely, for any vertex $v = (x_1 \dots x_d)$ in G_d, with $x_i \ge 0$ for any $1 \le i \le d$, we assign to v color $c(v)$ defined as follows :

(1) for any v such that $\sum_{i=1}^{d} x_i \equiv 0 \bmod 2$,

$$c(v) = [(\frac{1}{2}\sum_{i=1}^{d}(2i-1)qx_i) \; mod \; (2d-1)q + 1] + (p + (2d-1)q)$$

(2) for any v such that $\sum_{i=1}^{d} x_i \equiv 1 \bmod 2$,

$$c(v) = [(\frac{1}{2}[\sum_{i=1}^{d}(2i-1)qx_i - odd(q)]) \; mod \; (2d-1)q + 1$$

In order to prove that this is an $L(p, q)$ labeling of G_d, let us consider the two following cases :

- u and v are neighbors in G_d, thus they differ by one on exactly one coordinate x_i, $1 \le i \le d$. Let $u = (x_1, \dots x_i \dots x_d)$ and $v = (y_1, \dots y_i \dots y_d)$, $S(u) = \sum_{i=1}^{d} x_i$ and $S(v) = \sum_{i=1}^{d} y_i$. We clearly see that if $S(u)$ is even, then $S(v)$ is odd and vice-versa. Since in the case where $S(u)$ is even (resp. odd), $c(u) \in [p + (2d-1)q; p + (4d-2)q]$ (resp. $c(u) \in [0; (2d-1)q]$), we conclude that in all the cases, $|c(v) - c(u)| \ge p$.

- Suppose now that u and v lie at distance 2 in G_d : thus u and v differ on two coordinates x_i and x_j, $1 \le i \le j \le d$. W.l.o.g. we can consider only two cases, supposing $u = (x_1, \dots x_i \dots x_d)$: (a) $v = (x_1, \dots x_i + 1 \dots x_j + 1 \dots x_d)$ (where possibly $i = j$) and (b) $v = (x_1, \dots x_i + 1 \dots x_j - 1 \dots x_d)$ (where $i \neq j$). We also know that $S(u)$ and $S(v)$ have same parity. Thus for each of the cases (a) and (b), there are 2 cases to consider : (a-even) (resp. (b-even)) in case (a) (resp. (b)) when $S(u)$ and $S(v)$ are even and (a-odd) (resp. (b-odd)) in case (a) (resp. (b)) when $S(u)$ and $S(v)$ are odd. We detail each of those 4 cases below.

  - (a-even) : $|c(v) - c(u)| = \frac{1}{2}((2i-1)q + (2j-1)q)$, that is $|c(v) - c(u)| = (i + j - 1)q$. Hence, $c(u)$ and $c(v)$ differ by at least q, since $1 \le i, j \le d$.
  - (b-even) : Since $i \neq j$ let us suppose w.l.o.g. that $1 \le i < j \le d$. Hence $|c(u) - c(v)| = \frac{1}{2}((2j-1)q - (2i-1)q)$, that is $|c(u) - c(v)| = (j - i)q$. Since $1 \le i < j \le d$, $c(u)$ and $c(v)$ differ by at least q.

– (a-odd) : $|c(v) - c(u)| = \frac{1}{2}((2i-1)q + (2j-1)q)$, that is $|c(v) - c(u)| = (i+j-1)q$. But since $1 \leq i \leq j \leq d$, we conclude that $c(u)$ and $c(v)$ differ by at least q.

– (b-odd) : Since $i \neq j$ let us suppose w.l.o.g. that $1 \leq i < j \leq d$. Hence $|c(u) - c(v)| = \frac{1}{2}((2j-1)q - (2i-1)q)$, that is $|c(u) - c(v)| = (j-i)q$. But since $1 \leq i < j \leq d$, we conclude that $c(u)$ and $c(v)$ differ by at least q.

Hence, we have proved that the above mentioned coloring is an $L(p, q)$ labeling of the grid in the case $p, q, d \geq 1$ with $p \leq 2dq + 1$. Since it uses colors in the range $[0; (2d-1)q] \cup [p + (2d-1)q; p + (4d-2)q]$, we conclude that $\lambda_q^p(G_d) \leq p + (4d-2)q$. Altogether, this proves the lemma. □

As a consequence of Lemmas 1 and 2, we have the following theorem.

Theorem 1 ($L(p, q)$ labeling of d-dimensional Grids, for any value of $p, q, d \geq 1$) *Let $p \geq 1$ and $d \geq 1$. Then :*

- $2p + (2d-2)q \leq \lambda_q^p(G_d) \leq 2dq$ *when* $2 \leq 2p < q$
- $2p + (2d-2)q \leq \lambda_q^p(G_d) \leq 2p + (2d-1)q - 1$ *when* $1 \leq q \leq 2p \leq 4dq$
- $\lambda_q^p(G_d) = p + (4d-2)q$ *when* $p \geq 2dq + 1$

When $1 \leq q \leq 2p \leq 4dq$, the bounds we get coincide in the case $q = 1$, thus yielding an optimal $L(p, 1)$ labeling of G_d. We note that this generalizes Lemma 5 of [BPT00] and Theorem 3 of [DMP+02], and also answers an open problem stated in [DMP+02].

By combining some of the previous results, it is possible to improve some of the upper bounds obtained above. More precisely, this is done thanks to a combination of the results from Observation 1 and Theorem 1. This is the purpose of the following proposition.

Proposition 3 *For any $d \geq 1$:*

- *for any $q \geq 1$, and any $p = \alpha q$ with $1 \leq \alpha \leq 2d$, $\lambda_q^p(G_d) = 2p + (2d-2)q$*
- *for any $q \geq 1$ and any $p = \alpha q + \beta$ with $1 \leq \beta \leq q-1$ and $p \leq 2dq + \beta - q$, $\lambda_q^p(G_d) \leq 2p + 2dq - 2\beta$*

Proof : These two results derive from a combination of Observation 1 and Theorem 1. Suppose first that $p = \alpha q$, with $1 \leq \alpha \leq 2d$. By Theorem 1, we know that $\lambda_q^p(G_d) \geq 2p + (2d-2)q$. We are going to prove that $\lambda_q^p(G_d) \leq 2p + (2d-2)q$ as well. Indeed, by Observation 1, we know that $\lambda_q^p(G_d) \leq q \cdot \lambda_1^{\lceil \frac{p}{q} \rceil}(G_d)$, that is $\lambda_q^p(G_d) \leq q \cdot \lambda_1^{\alpha}(G_d)$. Since $1 \leq \alpha \leq 2d$, we know by Theorem 1 that $\lambda_1^{\alpha}(G_d) = 2\alpha + 2d - 2$. Hence, we conclude that $\lambda_q^p(G_d) \leq q(2\alpha + 2d - 2)$. But since $\alpha q = p$, we finally have $\lambda_q^p(G_d) \leq 2p + (2d-2)q$.

Now suppose that $q \geq 1$, and $p = \alpha q + \beta$ with $1 \leq \beta \leq q - 1$. Suppose also that $p \leq 2dq + \beta - q$. As previously, we apply Observation 1, which yields $\lambda_q^p(G_d) \leq q \cdot \lambda_1^{\lceil \frac{p}{q} \rceil}(G_d)$, that is $\lambda_q^p(G_d) \leq q \cdot \lambda_1^{\alpha+1}(G_d)$. However, since we suppose $p \leq 2dq + \beta - q$, this implies $\frac{p-\beta}{q} + 1 \leq 2d$, that is $\alpha + 1 \leq 2d$. Hence, we now apply Theorem 1, and obtain that $\lambda_1^{\alpha+1}(G_d) \leq 2(\alpha+1) + 2d - 2$. Thus, altogether, we obtain $\lambda_q^p(G_d) \leq q \cdot (2\alpha + 2d)$, that is $\lambda_q^p(G_d) \leq q \cdot (2(\frac{p-\beta}{q}) + 2d)$, ie $\lambda_q^p(G_d) \leq 2p + 2dq - 2\beta$. Hence the result. □

We note that in the first case of Proposition 3 above, we obtain the optimal value, while in second case we improve the results of Theorem 1 when $p, q, d \geq 1$, $\beta \geq \frac{q+2}{2}$ and $p \leq 2dq + \beta - q$.

We also note that in all the results presented above, when the bounds do not coincide, they differ by an additive factor at most equal to $\min\{q-1, 2(q-\beta)\}$ (where β is the rest of the division of p by q) when $1 \leq q \leq 2p \leq 4dq$, and equal to $2q - 2p \leq 2q - 2$ when $2 \leq 2p < q$.

Resp. values of p and q	$\lambda_q^p(G_d) \geq$	$\lambda_q^p(G_d) \leq$	Gap	No-hole Property
$p \neq 0$; $q = 0$	$\boldsymbol{p}$		0	No (except for $p = 1$)
$p = 0$; $q \neq 0$	$\boldsymbol{(2d-1)q}$		0	No (except for $q = 1$)
$p, q \geq 1$; $2p < q$	$2p + (2d-2)q$	$2dq$	$2q - 2p$	No
$q = 1$; $1 \leq p \leq 2d$	$\boldsymbol{2p + 2d - 2}$		0	Yes (Prop 4)
$p, q \geq 1$; $q \leq 2p \leq 4dq$	$2p + (2d-2)q$	$2p + (2d-1)q - 1$	$q - 1$	Yes (Prop 4)
$p, q \geq 1$; $p = \alpha q$; $\alpha \leq 2d$	$\boldsymbol{2p + (2d-2)q}$		0	No
$p, q \geq 1$; $p = \alpha q + \beta$ $1 \leq \beta \leq q-1$; $p \leq 2dq + \beta - q$	$2p + (2d-2)q$	$2p + 2dq - 2\beta$	$2(q - \beta)$	No
$p, q \geq 1$; $p \geq 2dq + 1$	$\boldsymbol{p + (4d-2)q}$		0	No

Table 1: $L(p,q)$ labeling of G_d : Summary of the results

Moreover, in the case $1 \leq q \leq 2p \leq 4dq$, for sufficiently large grids (that is, when the x_is are large enough for each $1 \leq i \leq d$), the coloring we propose to achieve an $L(p,q)$ labeling satisfies the *no-hole* property, that is all the colors in the range $[0; 2p + 2(d-1)q - 1]$ are used. This is the purpose of the following Proposition 4 below.

Proposition 4 (No-hole property of $L(p,q)$ labeling of G_d, when $1 \leq q \leq 2p \leq 4dq$) *Let $d \geq 1$. In that case, when $1 \leq q \leq 2p \leq 4dq$, there exists a no-hole $L(p,q)$ labeling of G_d such that $\lambda_q^p(G_d) = 2p + (2d-1)q - 1$.*

Proof : In Proof of Lemma 2, in the case $1 \leq q \leq 2p \leq 4dq$, we assign to any vertex $u = (x_1, x_2 \dots x_d)$ color $c(u) = (\sum_{i=1}^{d}(p + (i-1)q)x_i)$ *mod* $(2p + (2d-1)q)$. Let $U = \{u_k = (0, 0 \dots 0, 2k) | 1 \leq k \leq 2p + (2d-1)q\}$: in other words, in any vertex u_k, $1 \leq k \leq 2p + (2d-1)q$, only the d-th coordinate x_d is different from 0, and $x_d = 2k$. In that case, $c(u_k) \equiv (p + (d-1)q) \cdot 2k$ mod $2p + (2d-1)q$ by definition. However, $(p + (d-1)q) \cdot 2k = (k-1) \cdot (2p + (2d-1)q) + (2p + 2(d-1)q - k)$, hence $c(u_k) \equiv 2p + 2(d-1)q - k$, since $k \leq 2p + (2d-1)q$. Since the grid G_d is supposed to be sufficiently large, there is no restriction on the choice of x_d, and thus on the choice of k. Hence, one can see that each vertex u_k of U is assigned a unique color $c(u_k) = 2p + (2d-1)q - k$. Since k takes all the values between 1 and $2p + (2d-1)q$, we conclude that the vertices of U are assigned colors that take all the values in the range $[0; 2p + (2d-1)q - 1]$. Hence, the proposed labeling is a no-hole $L(p,q)$ labeling of G_d. □

Clearly, in the other cases, the proposed $L(p,q)$ labelings cannot be no-hole labelings, because some colors are forbidden. Indeed, in the case $2 \leq 2p < q$, colors are taken in the set $\{0, q, 2q, \dots 2dq\}$, thus it cannot be a no-hole coloring. In the case $p \geq 2dq + 1$, the colors ranging in the interval $[(2d-1)q + 1; p + (2d-1)q - 1]$ are forbidden, thus the coloring we suggest cannot be no-hole.

Table 1 summarizes the results obtained in Section 2 concerning bounds for the $L(p,q)$ coloring of G_d, for all the possible cases. In this table, we give the lower and upper bounds for $\lambda_q^p(G_d)$; they are given in bold characters when the bounds coincide. We also mention the gap between the upper and lower bounds when they do not coincide. Finally, in the rightmost column, we state whether the no-hole property holds for the colorings suggested in this paper.

We also mention that the upper bounds we obtain here concerning the $L(p,q)$ labeling of G_d are also upper bounds for the $L(p,q)$ labeling of hypercubes of dimension d, H_d. Indeed, H_d is isomorphic to the grid G_d where each x_i can take only two values (Hence, there are only two vertices lying in each dimension of the grid). $L(2,1)$ labeling of the hypercube of dimension d has been studied in [GY92], where the authors proved that $d + 3 \leq \lambda_1^2(H_d) \leq 2d + 1$, and where it was conjectured that $\lambda_1^2(H_d) = d+3$. The upper bound has been improved later in cite [WGM95] thanks to a technique coming from coding theory. However, to our knowledge, the $L(p,q)$ labeling of H_d has not been studied for general p and q. Hence, the results stated in Corollary 1 below constitute a first approach for tackling this problem. However, these upper bounds need to be improved.

Corollary 1 ($L(p,q)$ labeling of d-dimensional Hypercubes, for any value of $p, q, d \geq 1$) *Let $p \geq 1$ and $d \geq 1$. Then :*

- $\lambda_0^p(H_d) = p$ *when* $p, d \geq 1$
- $\lambda_q^0(H_d) \leq (2d-1)q$ *when* $q, d \geq 1$
- $\lambda_q^p(H_d) \leq 2dq$ *when* $2 \leq 2p < q$
- $\lambda_q^p(H_d) \leq 2p + (2d-1)q - 1$ *when* $1 \leq q \leq 2p \leq 4dq$
- $\lambda_q^p(H_d) \leq 2p + (2d-2)q$ *when* $p = \alpha q$ *with* $p, q \geq 1$ *and* $\alpha \leq 2d$
- $\lambda_q^p(H_d) \leq 2p + 2dq - 2\beta$ *when* $p, q \geq 1$, $p = \alpha q + \beta$ *with* $1 \leq \beta \leq q-1$ *and* $p \leq 2dq + \beta - q$
- $\lambda_q^p(H_d) \leq p + (4d-2)q$ *when* $p \geq 2dq + 1$

We note that we get the equality $\lambda_0^p(H_d) = p$ in the case $q = 0$ since we need to have a gap of at least p between any vertex colored 0 and its neighbor.

We finish this section by mentioning a result for the labeling of G_d with k constraints, each equal to 1. This labeling is usually denoted as an $L(\vec{1}_k)$ labeling. The results we give here are derived from a study initiated in [FGR03]. This is the purpose of the remark below.

Remark 1 ($L(\vec{1}_k)$ labeling of G_d) *In [FGR03], the authors have considered the* k-distance coloring *in the d-dimensional grid G_d. k-distance coloring of a graph G is a coloring of its vertices such that two vertices lying at distance less than or equal to k must be assigned different colors. Clearly, k-distance coloring is equivalent to the $L(\vec{1}_k)$ labeling, that is the $L(1, 1 \ldots 1)$ labeling with k constraints on the distances, each equal to 1).*

In their paper, the authors prove that $\lambda_1^1(G_d) = 2d$ for any $d \geq 1$. This result appears as a particular result of Theorem 1, when $p = q = 1$.

Another result from [FGR03] addresses the $L(\vec{1}_k)$ labeling problem in the 2-dimensional grid G_2, for any $k \geq 1$. Their result is the following :

- *if k is even, then* $\lambda(\vec{1}_k)[G_2] = \frac{(k+1)^2+1}{2} - 1$ *;*
- *if k is odd, then* $\lambda(\vec{1}_k)[G_2] = \frac{(k+1)^2}{2} - 1$.

We note that this result has also been independently given in [BPT02].

3 Conclusion

In this paper, we have addressed the frequency assignment problem with constraints on the distances. We have first given general bounds for the λ number when k constraints are given for the k first distances.

We have also addressed the problem of the $L(p,q)$ labeling in d-dimensional grids. These results are optimal in the cases $p = 0$, $q = 0$, $p \geq 2dq+1$, $p = \alpha q$ with $1 \leq \alpha \leq 2d$, and also in the case $q = 1$ (where in the latter case, our result answers an open question from [DMP+02], and generalizes results from [DMP+02] and [BPT00]). The only case where the result is not optimal is when p is not a multiple of q (thus, $p = \alpha \cdot q + \beta$, with $1 \leq \beta \leq q-1$) and $2 \leq 2p \leq 4dq$; in that case, the lower and upper bounds for $\lambda_q^p(G_d)$ differ by $\min\{q-1, 2(q-\beta)\}$ (in the case $2p \geq q$), or by $2q-2p \leq 2q-2$ (in the case $2p < q$). We proved that the coloring of the vertices we propose in the case $1 \leq q \leq 2p \leq 4dq$, though not necessarily optimal (when $q \geq 2$), is a no-hole coloring. We have also derived some upper bounds for the $L(p,q)$ coloring of d-dimensional hypercubes.

Finally, we wish to end this paper by suggesting that, using similar techniques, the results presented here could be extended to the $L(p,q,r)$ labeling problem in d-dimensional grids G_d.

References

[BKTvL00] H.L. Bodlaender, T. Kloks, R.B. Tan, and J. van Leeuwen. λ-coloring of graphs. In *Proc. STACS 2000 : 17th Annual Symposium on Theoretical Aspect of Computer Science*, volume 1770, pages 395–406. Lecture Notes Computer Science, Springer-Verlag Berlin, 2000.

[BPT00] A.A Bertossi, C.M. Pinotti, and R.B. Tan. Efficient use of radio spectrum in wireless networks with channel separation between close stations. In *Proc. DIAL M for Mobility 2000 (Fourth International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications)*, 2000.

[BPT02] A.A Bertossi, C.M. Pinotti, and R.B. Tan. Channel assignment with separation for special classes of wireless networks : Grids and rings. In *Proc. IPDPS'02 (International Parallel and Distributed Processing Symposium)*, pages 28–33. IEEE Computer Society, 2002.

[CK96] G.J. Chang and D. Kuo. The $L(2,1)$-labeling on graphs. *SIAM J. Discrete Math.*, 9:309–316, 1996.

[CKK+02] G.J. Chang, W.-T. Ke, D. Kuo, D.D.-F. Liu, and R.K. Yeh. On $L(d,1)$-labelings of graphs. *Discrete Mathematics*, 220:57–66, 2002.

[CP01] T. Calamoneri and R. Petreschi. $L(2,1)$-labeling of planar graphs. In *Proc. DIAL M for Mobility 2001 (Fifth International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications)*, pages 28–33, 2001.

[DMP+02] A. Dubhashi, S. MVS, A. Pati, S. R., and A.M. Shende. Channel assignment for wireless networks modelled as d-dimensional square grids. In *Proc. IWDC'02 (International Workshop on Distributed Computing)*, volume 2571, pages 130–141. Lecture Notes in Computer Science - Springer Verlag, 2002.

[FGR03] G. Fertin, E. Godard, and A. Raspaud. Acyclic and k-distance coloring of the grid. *Information Processing Letters*, 87(1):51–58, 2003.

[FKK01] J. Fiala, T. Kloks, and J. Kratochvil. Fixed-parameter complexity of λ-labelings. *Discrete Applied Mathematics*, 113:59–72, 2001.

[FKP01] J. Fiala, J. Kratochvil, and A. Proskurowski. Distance constrained labelings of precolored trees. In *Proc. 7th Italian Conference, ICTCS 2001*, volume 2202, pages 285–292. Lecture Notes in Computer Science - Springer Verlag, 2001.

[GY92] J.R. Griggs and R.K. Yeh. Labeling graphs with a condition at distance two. *SIAM J. Discrete Math.*, 5:586–595, 1992.

[Hal80] W.K. Hale. Frequency assignment : theory and applications. *Proc. IEEE*, 60:1497–1514, 1980.

[Jha00] P.K. Jha. Optimal $L(2,1)$-labeling of cartesian products of cycles with an application to independent domination. *IEEE Trans. Circuits & Systems I: Fundamental Theory and Appl.*, 47:1531–1534, 2000.

[JNS+00] P.K. Jha, A. Narayanan, P. Sood, K. Sundaram, and V. Sunder. On $L(2,1)$-labeling of the cartesian product of a cycle and a path. *Ars Combin.*, 55:81–89, 2000.

[Kra] D. Kral. Coloring powers of chordal graphs. *SIAM J. Discr. Math.* To appear.

[MS02] M. Molloy and M.R. Salavatipour. Frequency channel assignment on planar networks. In *In Proc. 10th Annual European Symposium (ESA 2002), Rome, Italy, September 2002*, volume 2461, pages 736–747. Lecture Notes Computer Science, Springer-Verlag Berlin, 2002.

[Sak94] D. Sakai. Labeling chordal graphs with a condition at distance two. *SIAM J. Discrete Math.*, 7:133–140, 1994.

[WGM95] M. Whittlesey, J. Georges, and D.W. Mauro. On the λ-number of Q_n and related graphs. *SIAM J. Discrete Math.*, 8:499–506, 1995.