

# $L(p, q)$ Labeling of $d$ -Dimensional Grids (Extended Abstract)

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## Abstract

In this paper, we address the problem of  $\lambda$  labelings, concerning frequency assignment for telecommunication networks. In this model, stations within a given radius  $r$  must use frequencies that differ at least by a value  $p$ , while stations that are within a larger radius  $r' > r$  must use frequencies that differ by at least another value  $q$ . The aim is to minimize the span of frequencies used in the network. This can be modeled by a graph coloring problem, called the  $L(p, q)$  labeling, where one wants to label vertices of the graph  $G$  modeling the network by integers in the range  $[0; M]$ , while minimizing the value of  $M$ .  $M$  is then called the  $\lambda$  number of  $G$ , and is denoted by  $\lambda_q^p(G)$ .

We study here the  $L(p, q)$  labeling problem, where, for a specific class of networks, namely the  $d$ -dimensional grid  $G_d = G[n_1, n_2 \dots n_d]$ , we give bounds on the value of the  $\lambda$  number of an  $L(p, q)$  labeling for any  $p, q \geq 0$  in any  $d$ -dimensional grid  $G_d$ . Some of these results are optimal (namely, in the following cases: (1)  $p = 0$ , (2)  $q = 0$ , (3)  $q = 1$  and (4)  $p \geq 2dq + 1$ ); when the results we obtain are not optimal, we observe that the bounds differ by an additive factor never exceeding  $2q - 2$ . The optimal result we obtain in the case  $q = 1$  answers an open problem stated by Dubhashi et al. [DMP<sup>+</sup>02], and generalizes results from [BPT00] and [DMP<sup>+</sup>02].

## 1 Introduction

In this paper, we study the *frequency assignment problem*, originally introduced in [Hal80], where radio transmitters that are geographically close may interfere if they are assigned close frequencies. This problem arises in mobile or wireless networks. Generally, this problem is modeled by a graph coloring problem, where the transmitters are the vertices, and an edge joins two transmitters that are sufficiently close to potentially interfere. The aim here is to color (i.e. give a value, corresponding to the frequency) the vertices of the graph in such a way that:

- any two neighbors (transmitters that are very close) are assigned colors (frequencies) that differ by a parameter at least  $p$  ;
- any two vertices at distance 2 (transmitters that are close) are assigned colors (frequencies) that differ by a parameter at least  $q$  ;
- the greatest value for the colors is minimized.

It has been proved that under this model, we could assume the colors to be integers, starting at 0 [GY92]. In that case, the minimum range of frequencies that is necessary to assign to the vertices of a graph  $G$  is denoted  $\lambda_q^p(G)$ , and the problem itself is usually called the  $L(p, q)$  labeling problem. The frequency assignment problem has been studied in many different specific topologies [GY92, Sak94, WGM95, BPT00, BKTvL00, CKK<sup>+</sup>02, MS02, BPT02]. The case  $p = 2$  and  $q = 1$  is the most widely studied (see for instance [CK96, JNS<sup>+</sup>00, Jha00, CP01]). Some variants of the model also exist, such as the following generalization where one gives  $k$  constraints on the  $k$  first distances (any two vertices at distance  $1 \leq i \leq k$  in  $G$  must be assigned colors differing by at least  $\delta_i$ ). One of the issues also considered in the frequency assignment problem is the *no-hole property*, where one wants to know whether a given coloring uses all the possible colors in the range  $[0; \lambda_q^p(G)]$ .

In this paper, we mainly focus on the  $L(p, q)$  labeling problem. More precisely, in Section 2 we study the case of the  $L(p, q)$  labeling in the  $d$ -dimensional grid  $G_d$ . We first address in Section 2.1 the cases where  $p = 0$  or  $q = 0$ . In Section 2.2, we give results for the  $L(p, q)$  labeling of  $G_d$  for any  $p, q, d \geq 1$ . We give lower and upper bounds on  $\lambda_q^p(G_d)$ , and show that in some cases, these bounds coincide. Notably, in the case  $q = 1$ , the results we obtain are optimal ; this answers an open problem stated by Dubhashi et al. in [DMP<sup>+</sup>02], and generalizes results from [BPT00] and [DMP<sup>+</sup>02]. The results we give are also optimal when  $p = 0, q = 0$ , and  $p \geq 2dq + 1$ . We also prove that in some cases (namely, when  $1 \leq p \leq 2dq$ ), the coloring we propose satisfies the no-hole property.

## 2 $L(p, q)$ labeling of $G_d$

We study here the  $\lambda$  labeling problem with two constraints on the distances, in a particular network topology, namely the  $d$ -dimensional grid  $G_d = G[n_1, n_2 \dots n_d]$ . We first address the special cases  $p = 0$  (resp.  $q = 0$ ) in Section 2.1. We then address the more general case where  $p, q \geq 1$  in Section 2.2.

Due to space limitations, only a very restricted number of proofs will be given here. However, we note that they very often rely on the same process, i.e. (1) a deep study of the constraints induced by the grid topology, in order to derive lower bounds for  $\lambda_q^p(G_d)$ , and (2) a specific coloring for any vertex  $u \in V(G_d)$ , for which we show that it is a valid  $L(p, q)$  labeling of  $G_d$  (this gives an upper bound for  $\lambda_q^p(G_d)$ ).

### 2.1 $L(p, q)$ labeling when $p = 0$ or $q = 0$

**Property 1** For any  $p \geq 0$  and  $d \geq 1$ ,  $\lambda_0^p(G_d) = p$ .

**Property 2** For any  $q \geq 0$  and  $d \geq 1$ ,  $\lambda_q^0(G_d) = (2d - 1)q$ .

We note that except in specific cases, the colorings we have given above do not satisfy the no-hole property (we recall that the no-hole property holds when all colors in the range  $[0; \lambda_q^p]$  are used). Indeed, in the case  $q = 0$ , only colors 0 and  $p$  are used, thus the coloring cannot be no-hole, except when  $p = 1$ . Similarly, in the case  $p = 0$ , the colors used are taken in the set  $\{0, q, 2q \dots (2d - 1)q\}$ , thus the coloring cannot be no-hole, except in the specific case  $q = 1$ .

## 2.2 $L(p, q)$ labeling when $p, q \geq 1$

We now address the  $L(p, q)$  labeling of  $G_d$ , for any values of  $p, q \geq 1$  and  $d \geq 1$ . First, we note that we can obtain two trivial upper bounds on  $\lambda_q^p(G_d)$ . The first one relies on an existing  $L(p', 1)$  labeling of  $G_d$ .

**Observation 1** For any  $p, q, d \geq 1$ ,  $\lambda_q^p(G_d) \leq q \cdot \lambda_1^{\lceil \frac{p}{q} \rceil}(G_d)$ .

There exists another upper bound for  $\lambda_q^p(G_d)$ , that relies on an existing  $L(1, 1)$  labeling of  $G_d$ .

**Observation 2** For any  $p, q, d \geq 1$ ,  $\lambda_q^p(G_d) \leq \max\{p, q\} \cdot 2d$ .

The two above mentioned simple observations present the disadvantage to be based upon an existing labeling of  $G_d$ . In the following, we study the problem in more details, and define upper and lower bounds on  $\lambda_q^p(G_d)$  for all values of  $p, q, d \geq 1$ . These results are summarized in Theorem 1.

**Theorem 1 ( $L(p, q)$  labeling of  $G_d$ , for any value of  $p, q, d \geq 1$ )** Let  $p \geq 1$  and  $d \geq 1$ . Then:

- (1)  $2p + (2d - 2)q \leq \lambda_q^p(G_d) \leq 2dq$  when  $2 \leq 2p < q$
- (2)  $2p + (2d - 2)q \leq \lambda_q^p(G_d) \leq 2p + (2d - 1)q - 1$  when  $1 \leq q \leq 2p \leq 4dq$
- (3)  $\lambda_q^p(G_d) = p + (4d - 2)q$  when  $p \geq 2dq + 1$

**Proof:** Due to space limitations, we only give the proof for Case 2, that is when  $1 \leq q \leq 2p \leq 4dq$ . We first prove the lower bound of  $2p + (2d - 2)q$ : suppose that it is possible to  $L(p, q)$  label the vertices of  $G_d$  with  $M$  colors, with  $M \leq 2p + (2d - 2)q - 1$ . We will first show that in that case, no vertex of degree  $2d$  in  $G_d$  can be assigned a color in the range  $[p - 1; p + (2d - 1)q - 1]$ . Indeed, suppose there exists a vertex  $u \in V(G_d)$  such that  $u$  is assigned color  $p + x$ , with  $-1 \leq x \leq (2d - 1)q - 1$ . Then, all its neighbors must be assigned a color in the range  $[0; x] \cup [2p + x; M]$ , because of the gap of at least  $p$  that must exist between neighbors. Within this range, one must be able to get  $2d$  values, each pair of which differs of at least  $q$ . Let us distinguish two cases: (i)  $x = -1$  and (ii)  $x \geq 0$ . In case (i), it is clear that all the colors must be in the range  $[2p - 1; M]$ . In other words, if we want to be able to assign the  $2d$  colors of the neighbors, we must have  $2p - 1 + (2d - 1)q \leq M$ . However, we supposed  $M \leq 2p + (2d - 2)q - 1$ , hence the contradiction since  $q \geq 1$ . Now suppose that (ii)  $x \geq 0$ ; we distinguish two more cases: (ii-1)  $x = kq$  and (ii-2)  $x = kq - i$ , with  $1 \leq i \leq q - 1$ . In case (ii-1), we can use  $(k + 1)$

colors in the range  $[0; kq]$  (more precisely, colors  $0, q, 2q \dots kq$ ). Hence there remains  $2d - (k - 1)$  to get in the range  $[2p + x; M]$ . For this, we must have  $2p + x + (2d - (k - 1) - 1)q \leq M$ . This gives  $2p + (2d - 2)q \leq M$ , a contradiction. In case (ii-2), only  $k$  colors can be assigned in the range  $[0; x]$ . Thus  $2d - k$  colors must be assigned in the range  $[2p + x; M]$ , which can be the case only if  $2p + x + (2d - k - 1)q \leq M$ . This can happen only when  $i \geq q + 1$ , a contradiction too. Thus we conclude that if  $\lambda_q^p(G_d) = M$ , no vertex of degree  $2d$  in  $G_d$  can be assigned a color in the range  $[p - 1; p + (2d - 1)q - 1]$ .

In other words, if such a coloring exists, all vertices of degree  $2d$  are assigned colors in the range  $[0; p - 2] \cup [p + (2d - 1)q; M]$ . Let  $I_1 = [0; p - 2]$  and  $I_2 = [p + (2d - 1)q; M]$ , with  $M = 2p + (2d - 2)q - j$ ,  $j \geq 1$ . Clearly,  $I_1$  contains  $p - 1$  integers, and  $I_2$  contains  $p - q - j + 1 < p$  integers (since  $j, q \geq 1$ ). This means that if a vertex  $u$  of degree  $2d$  in  $G_d$  is assigned a color in  $I_1$  (resp.  $I_2$ ), all its neighbors must be assigned colors in  $I_2$  (resp.  $I_1$ ) – supposing that all the neighbors of  $u$  are of degree  $2d$ , which is the case if  $G_d$  is “big” enough. However, in order for  $I_1$  (resp.  $I_2$ ) to support  $2d$  colors that, pairwise, admit a gap of  $q$ , the two following conditions must be fulfilled: (1)  $(2d - 1)q \leq p - 2$  and (2)  $p + (2d - 1)q + (2d - 1)q \leq M$ . In other words, we must have (1')  $p \geq (2d - 1)q + 2$  and (2')  $p \geq 2dq + j$ . Since  $j, q \geq 1$ , condition (2') implies condition (1'). Thus, in order to have a valid  $L(p, q)$  labeling with  $\lambda_q^p(G_d) = M$ , we must have  $p \geq 2dq + j$  with  $j \geq 1$ . However, we supposed  $p \leq 2dq$ , hence the contradiction.

Now we prove the upper bound of  $2p + (2d - 1)q - 1$ : for any vertex  $v = (x_1 \dots x_d)$  in  $G_d$ , with  $x_i \geq 0$  for any  $1 \leq i \leq d$ , we assign to  $v$  color  $c(v)$  defined as follows:

$$c(v) = \left( \sum_{i=1}^d (p + (i - 1) \cdot q)x_i \right) \text{ mod } (2p + (2d - 1)q)$$

We are going to prove that this coloring is an  $L(p, q)$  labeling of  $G_d$ . For this, we distinguish two cases:

- $u$  and  $v$  are neighbors in  $G_d$ , thus they differ on one coordinate  $x_i$ ,  $1 \leq i \leq d$ . W.l.o.g., suppose  $u = (x_1, \dots, x_i \dots x_d)$  and  $v = (x_1, \dots, x_i + 1 \dots x_d)$ . Thus  $|c(v) - c(u)| = (p + (i - 1)q)$ . Since  $1 \leq i \leq d$ , we have that  $|c(v) - c(u)| \geq p$ .
- $u$  and  $v$  lie at distance 2 in  $G_d$ , thus they differ on two coordinates  $x_i$  and  $x_j$ ,  $1 \leq i \leq j \leq d$ . W.l.o.g. we can consider only two cases, supposing  $u = (x_1, \dots, x_i \dots x_j \dots x_d)$ : (1)  $v = (x_1, \dots, x_i + 1 \dots x_j + 1 \dots x_d)$  (where possibly  $i = j$ ) and (2)  $v = (x_1, \dots, x_i + 1 \dots x_j - 1 \dots x_d)$  (where  $i \neq j$ ). In case (1),  $|c(u) - c(v)| = 2p + (i + j - 2)q$ . In that case,  $|c(u) - c(v)| \geq q$ , except maybe when  $i = j = 1$ . However, when  $i = j = 1$ , then  $|c(u) - c(v)| = 2p$ , and by hypothesis we know that  $2p \geq q$ . Thus  $|c(u) - c(v)| \geq q$  in all the cases. In case (2), let us suppose w.l.o.g. that  $j > i$  (we recall we cannot have  $j = i$ ). Then we have  $|c(u) - c(v)| = (j - i)q$ . Thus, for any two vertices  $u, v \in V(G_d)$  that lie at distance 2, we have  $|c(u) - c(v)| \geq q$ .

Hence, we have proved that the above mentioned coloring is an  $L(p, q)$  labeling of the grid in the case  $p, q, d \geq 1$ ,  $1 \leq q \leq 2p \leq 4dq$ . Since it uses colors in the range  $[0; 2p + (2d - 1)q - 1]$ ,

Values of $p$ and $q$	$\lambda_q^p(G_d) \geq$	$\lambda_q^p(G_d) \leq$	Gap	No-hole
$p \neq 0 ; q = 0$	<b><math>p</math></b>		0	No
$p = 0 ; q \neq 0$	<b><math>(2d - 1)q</math></b>		0	No
$p, q \geq 1 ; 2p < q$	$2p + (2d - 2)q$	$2dq$	$2q - 2p$	No
$q = 1 ; 1 \leq p \leq 2d$	<b><math>2p + 2d - 2</math></b>		0	Yes (Prop 1)
$p, q \geq 1 ; q \leq 2p \leq 4dq$	$2p + (2d - 2)q$	$2p + (2d - 1)q - 1$	$q - 1$	Yes (Prop 1)
$p, q \geq 1 ; p \geq 2dq + 1$	<b><math>p + (4d - 2)q</math></b>		0	No

Table 1:  $L(p, q)$  labeling of  $G_d$ : Summary of the results

we conclude that  $\lambda_q^p(G_d) \leq 2p + (2d - 1)q - 1$ . □

When  $1 \leq q \leq 2p \leq 4dq$ , the bounds we get coincide in the case  $q = 1$ , thus yielding an optimal  $L(p, 1)$  labeling of  $G_d$ . We note that this generalizes Lemma 5 of [BPT00] and Theorem 3 of [DMP<sup>+</sup>02], and also answers an open problem stated in [DMP<sup>+</sup>02]. We also note that when the above bounds do not coincide, they differ by an additive factor equal to  $q - 1$  when  $1 \leq q \leq 2p \leq 4dq$ , and equal to  $2q - 2p \leq 2q - 2$  when  $2 \leq 2p < q$ .

Moreover, in the case  $1 \leq q \leq 2p \leq 4dq$ , for sufficiently large grids (that is, when the  $x_i$ s are large enough for each  $1 \leq i \leq d$ ), the coloring we propose to achieve an  $L(p, q)$  labeling satisfies the *no-hole* property, that is all the colors in the range  $[0; 2p + 2(d - 1)q - 1]$  are used. This is the purpose of the following Proposition 1 below.

**Proposition 1 (No-hole property when  $1 \leq q \leq 2p \leq 4dq$ )** *Let  $d \geq 1$ . In that case, when  $1 \leq q \leq 2p \leq 4dq$ , there exists a no-hole  $L(p, q)$  labeling of  $G_d$  such that  $\lambda_q^p(G_d) = 2p + (2d - 1)q - 1$ .*

In the other cases, the proposed  $L(p, q)$  labelings cannot be no-hole labelings, because some colors are forbidden. Indeed, in the case  $2 \leq 2p < q$ , colors are taken in the set  $\{0, q, 2q, \dots, 2dq\}$ , thus it cannot be a no-hole coloring. In the case  $p \geq 2dq + 1$ , the colors ranging in the interval  $[(2d - 1)q + 1; p + (2d - 1)q - 1]$  are forbidden, thus the coloring we suggest cannot be no-hole.

Table 1 summarizes the results obtained in Section 2 concerning bounds for the  $L(p, q)$  coloring of  $G_d$ , for all the possible cases. In this table, we give the lower and upper bounds for  $\lambda_q^p(G_d)$ ; they are given in bold characters when the bounds coincide. We also mention the gap between the upper and lower bounds, when they do not coincide. Finally, in the rightmost column, we state whether the no-hole property holds for the colorings suggested in this paper.

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