# No-Hole $L(p,0)$ Labelling of Cycles, Grids and Hypercubes*

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**Abstract.** In this paper, we address a particular case of the general problem of $\lambda$ labellings, concerning frequency assignment for telecommunication networks. In this model, stations within a given radius $r$ must use frequencies that differ at least by a value $p$, while stations that are within a larger radius $r' > r$ must use frequencies that differ by at least another value $q$. The aim is to minimize the span of frequencies used in the network. This can be modelled by a graph labelling problem, called the $L(p,q)$ labelling, where one wants to label vertices of the graph $G$ modelling the network by integers in the range $[0; M]$, while minimizing the value of $M$. $M$ is then called the $\lambda$ number of $G$, and is denoted by $\lambda_q^p(G)$.

Another parameter that sometimes needs to be optimized is the fact that all the possible frequencies (i.e., all the possible values in the span) are used. In this paper, we focus on this problem. More precisely, we want that: (1) all the frequencies are used and (2) condition (1) being satisfied, the span must be minimum. We call this the *no-hole* $L(p,q)$ labelling problem for $G$. Let $[0; M']$ be this new span and call the $\nu$ number of $G$ the value $M'$, and denote it by $\nu_q^p(G)$.

In this paper, we study a special case of no-hole $L(p,q)$ labelling, namely where $q = 0$. We also focus on some specific topologies: cycles, hypercubes, 2-dimensional grids and 2-dimensional tori. For each of the mentioned topologies cited above, we give bounds on the $\nu_0^p$ number and show optimality in some cases. The paper is concluded by giving new results concerning the (general, i.e. not necessarily no-hole) $L(p,q)$ labelling of hypercubes.

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## 1 Introduction

In this paper, we study the *frequency assignment problem*, that arises in wireless communication systems. We are interested here in minimizing the number of frequencies used in the framework where radio transmitters that are geographically close may interfere if they are assigned close frequencies. This problem has originally been introduced in [12] and later developed in [9], where it has been shown to be equivalent to a graph labelling problem, in which the nodes represent the transmitters, and any edge joins two transmitters that are sufficiently close to potentially interfere. The aim here is to label the nodes of the graph in such a way that:

- any two neighbours (transmitters that are very close) are assigned labels (frequencies) that differ by a parameter at least $p$ ;
- any two vertices at distance 2 (transmitters that are close) are assigned labels (frequencies) that differ by a parameter at least $q$ ;
- the greatest value for the labels is minimized.

It has been proved that under this model, we could assume the labels to be integers, starting at 0 [8]. In that case, the minimum range of frequencies that is necessary to assign to the vertices of a graph $G$ is denoted $\lambda_q^p(G)$, and the problem itself is usually called the $L(p, q)$ labelling problem. The frequency assignment problem has been studied in many different specific topologies [8, 14, 17, 1, 3, 5, 13, 2, 15, 16]. The case $p = 2$ and $q = 1$ is the most widely studied (see for instance [6, 11, 10, 4]). Some variants of the model also exist, such as the following generalization where one gives $k$ constraints on the $k$ first distances (any two vertices at distance $1 \leq i \leq k$ in $G$ must be assigned labels differing by at least $\delta_i$). One of the issues also considered in the frequency assignment problem is the *no-hole labelling*, where one wants to use all the frequencies in the span. More precisely, we want that: (1) all the frequencies are used and (2) condition (1) being satisfied, the span must be minimum. Let $[0; M']$ be the span of frequencies that we obtain. We then call the $\nu$ number of $G$ the value $M'$, and we denote it by $\nu_q^p(G)$. We note that depending on the values of $p$, $q$ and on the considered graph $G$, a no-hole labelling might not exist. In that case, we let $\nu_q^p(G) = \infty$. Hence, we clearly have $\nu_q^p(G) \geq \lambda_q^p(G)$ for any $p$, $q$ and $G$.

In this paper, we study a special case of no-hole $L(p, q)$ labelling, namely where $q = 0$. We also focus on some specific topologies: cycles, $d$-dimensional hypercubes, 2-dimensional grids and 2-dimensional tori. For each of the mentioned topologies cited above, we give bounds on the $\nu_0^p$ number for any value of $p \geq 1$ and $d \geq 1$, and show optimality in some cases. We conclude the paper with new results concerning the (general, i.e. not necessarily no-hole) $L(p, q)$ labelling of the $d$-dimensional hypercube, $H_d$.

## 2 No-Hole $L(p, 0)$ labellings

**Proposition 1 (General graphs).** *For any $p \geq 1$ and any connected graph $G$, if a no-hole labelling of $G$ exists, then:*

– $\nu_0^p(G) \geq 2p - 1$ *if* $G$ *is bipartite.*
– $\nu_0^p(G) \geq 2p$ *if* $G$ *is not bipartite.*

*Proof.* In any graph such that a no-hole $L(p, 0)$ labelling exists, there must, by definition, exist at least one vertex of label 0. Let $u$ be this vertex. Then all the neighbours of $u$ must be labelled at least $p$. Since the labelling is no-hole, all the labels in the range $[0; p]$ must be used. This is true, in particular, for label $p-1$. Let $v$ be a vertex whose label is $p - 1$. Then $v$ has all its neighbours labelled at least $2p - 1$.

Now, let $G$ be non bipartite, and suppose that $\nu_0^p(G) \leq 2p-1$. $G$ has at least one odd cycle. Consider the vertices on this cycle, and let $i \geq 0$ be the minimum label among them, assigned to vertex $v$. If $i \geq p$, then the neighbours of $v$, not being of minimum label, must be assigned a label at least $2p$, a contradiction. Hence, $i \in [0; p - 1]$. In that case, the two neighbours of $v$ on the cycle, say $w_1$ and $w_2$, are assigned labels at least $p + i$, that is in the range $[p; 2p - 1]$. But the neighbours of $w_1$ and $w_2$ on the cycle must be assigned labels in the range $[0; p - 1]$, etc. If we repeat this argument, we see that, when we will close the cycle, since it is odd, we will end up with a vertex $z$ whose two neighbours, say $x$ and $y$, are such that $x$ is assigned a label in the range $[0; p - 1]$, while $y$ is assigned a label in the range $[p; 2p-1]$. As $\nu_0^p(G) \leq 2p-1$, there is no possibility to label $z$ in an $L(p, 0)$ fashion, a contradiction.

**Observation 1** *For any graph* $G$ *of order* $n$ *that admits a no-hole labelling,* $n \geq \nu_0^p(G) + 1$

*Proof.* Suppose a no-hole labelling for $G$ exists. In order to be able to assign the vertices of $G$ all labels in the range $[0; \nu_0^p(G)]$, we must have $n \geq \nu_0^p(G) + 1$.

**Proposition 2 (Cycles).** *For any* $p \geq 1$ *and any graph* $G$*:*

– $\nu_0^p(C_n) = 2p$ *for any odd* $n \geq 2p + 1$
– $\nu_0^p(C_n) = 2p - 1$ *for any even* $n \geq 2p + 2$

*Proof.* First, suppose that $n$ is even. According to Proposition 1, we know that, if a no-hole $L(p, 0)$ labelling for $C_n$ exists, then $\nu_0^p(C_n) \geq 2p-1$. By Observation 1, we must have $n \geq 2p$. However, let $v$ be the vertex which is assigned label $p$. In that case, both its neighbours must be assigned label 0, because only labels in the range $[0; 2p - 1]$ are allowed, and the gap between two neighbours is at least $p$. Hence, $n \geq 2p + 1$; but since $n$ is even, we have $n \geq 2p + 2$. Suppose the vertices of $C_n$ are numbered clockwise from 1 to $n$. We give the following labelling function $c$ on $C_n$ : (a) for any vertex $v$ numbered $2i + 1$ $(0 \leq i \leq p-1)$, $c(v) = p + i$ ; (b) for any vertex $v$ numbered $2i$ $(1 \leq i \leq p)$, $c(v) = i - 1$ ; (c) for any vertex $v$ numbered $2i+1$, $i \geq p$, $c(v) = 2p-1$; (d) for any vertex $v$ numbered $2i$ $(i \geq p + 1)$, $c(v) = 0$ (cf. for instance Figure 1 (right), where $n = 2p + 2$).
Now, we show that this assignment is an $L(p, 0)$ no-hole labelling of $C_n$, for any even $n \geq 2p + 2$. First, consider any two neighbouring vertices $j$ and $j + 1$, $1 \leq j \leq 2p - 1$. If $j$ is even $j = 2i$, then $c(j) = i - 1$, while $c(j + 1) = p + i$,

thus the gap of at least $p$ is satisfied. If $j$ is odd $j = 2i + 1$, then $c(j) = p + i$, while $c(j + 1) = i$, which also satisfies the $L(p, 0)$ condition. Now consider two neighbours $j$ and $j+1$, with $2p+1 \leq j \leq n-1$. If $j$ is even, then $c(j) = 0$ while $c(j + 1) = 2p - 1$, and if $j$ is odd, then $c(j) = 2p - 1$ while $c(j + 1) = 0$. There still remain two cases to consider: (1) Vertices 1 and $n$ and (2) vertices $2p$ and $2p + 1$. However, in case (1) we have $c(1) = p$ and $c(n) = 0$ , while in case (2) we have $c(2p) = p - 1$ and $c(2p + 1) = 2p - 1$.
Consequently, the $L(p, 0)$ condition is satisfied. There is no hole as from the definition of the labelling, one can see that all labels are used on vertices 0 to $2p$: odd vertices $2i + 1$, $0 \leq i \leq p - 1$ have labels from $p$ to $2p - 1$, while even vertices $2i$, $1 \leq i \leq p$ are assigned labels from 0 to $p - 1$.

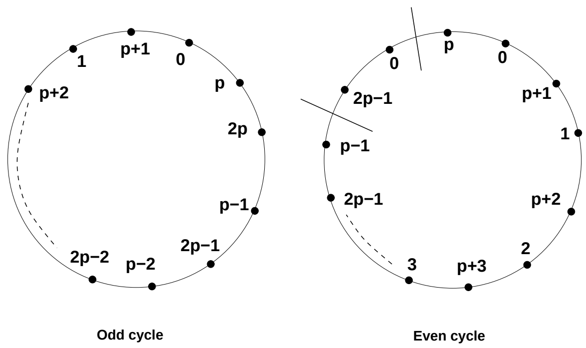


**Fig. 1.** $L(p, 0)$ no-hole labelling in cycles

Suppose that $n$ is odd. By Proposition 1, we know that, if a no-hole $L(p, 0)$ labelling for $C_n$ exists, then $\nu_0^p(C_n) \geq 2p$. Now, clearly, if $n \leq 2p-1$, then there is not enough vertices to use all the labels. Thus, we must have $n \geq 2p + 1$ in order that the no-hole labelling exists. Assume that $n \geq 2p+1$, and the vertices of $C_n$ are numbered clockwise from 1 to $n$. We define the following labelling function $c$ to $C_n$: (a) for any vertex $v$ numbered $2i+1$ $(0 \leq i \leq p)$, $c(v) = p-i$ ; (b) for any vertex $v$ numbered $2i$ $(1 \leq i \leq p)$, $c(v) = 2p - i + 1$ ; (c) for any vertex $v$ numbered $2i + 1$, $i \geq p+1$, $c(v) = 0$ ; (d) for any vertex $v$ numbered $2i$ $(i \geq p + 1)$, $c(v) = p$ (cf. for instance Figure 1 (left), where $n = 2p + 1$).

The labelling is an $L(p, 0)$ no-hole labelling of $C_n$, for any odd $n \geq 2p + 1$. First, consider any two neighbouring vertices $j$ and $j+1$, $1 \leq j \leq 2p$. If $j$ is even $j = 2i$, then $c(j) = 2p-i+1$, while $c(j+1) = p-i$, thus the gap of at least $p$ is satisfied. If $j$ is odd $j = 2i + 1$, then $c(j) = p - i$, while $c(j + 1) = 2p - i$, which also fulfills the $L(p, 0)$ condition. Now consider two neighbours $j$ and $j+1$, with $2p + 2 \leq j \leq n - 1$. If $j$ is even, then $c(j) = p$ while $c(j + 1) = 0$, and if $j$ is odd, then $c(j) = 0$ while $c(j+1) = p$. Now there remains some cases to consider: (1) $j = 2p + 1$ and $j + 1 = 2p + 2$ and (2) $j = n$ and $j + 1 = 0$ (that is, we "close" the cycle). But in both cases, we have $c(j) = 0$ and $c(j+1) = p$. Thus, altogether, the $L(p, 0)$ condition is satisfied. Now, by definition of the labelling, we can see that all the labels are used on vertices 0 to $2p$: vertices of the form $2i+1$, $0 \leq i \leq p$ are assigned labels from 0 to $p$, while vertices of the form $2i$, $1 \leq i \leq p$ are assigned labels from $p+1$ to $2p$.

**Proposition 3 (Hypercubes).** *For any $d-$dimensional hypercube $H_d$ such that $d \geq \frac{p+4}{2}$, $\nu_0^p(H_d) = 2p - 1$.*

**Sketch of Proof:** By Proposition 1, $\nu_0^p(H_d) \geq 2p - 1$. We will first show that $\nu_0^p(H_d) \leq 2p - 1$ (thus, proving the equality) for any $d \geq 2p - 1$ ; then, we will show that this result can be extended to any $d \geq \frac{p+4}{2}$. Suppose $d \geq 2p - 1$. The fact that $\nu_0^p(H_d) \leq 2p - 1$ is proved by homomorphism into the following graph $G'_p$: (a) the nodes of $G'_p$ are the integers between 0 and $2p-1$ and (b) there is an edge between $u$ and $v$ in $G'_p$ iff $|u - v| \geq p$. Clearly, $G'_p$ represents the constraints on the $L(p, 0)$ labelling, in the sense that any edge $(u, v)$ of $G'_p$ indicate that labels $u$ and $v$ can be assigned to neighboring nodes in $H_d$. We want to find an homomorphism $\mathcal{H}$ from $H_d$ to $G'_p$, i.e. we want to find a mapping from $V(H_d)$ to $V(G')$, where every node $v$ has an image $h(v)$ such that any edge $(u, v)$ in $H_d$ corresponds to an edge $(h(u), h(v))$ in $G'_p$. If we can do this, then we can find a labelling (more precisely, $c(v) = h(v)$ for any node $v$) that satisfies the $L(p, 0)$ constraints. Furthermore as this labelling has to be no-hole, we also need that every node of $G'_p$ is an image of at least one node of $H_d$. Let us partition the nodes of $H_d$ into $d+1$ sets: for any $0 \leq i \leq d$, the set $S_i$ corresponds to the nodes having $i$ bits equal to 0 in its binary coordinates. By definition of the hypercube, for every $0 \leq i \leq d$, $S_i$ is a stable set. In other words, all edges appear between different $S_i$s. More precisely, all the edges of $H_d$ appear between an $S_i$ and an $S_{i+1}$. Let us define the homomorphism so that all nodes belonging to the same $S_i$ have the same image by $\mathcal{H}$. Let $h_i$ be the image by $\mathcal{H}$ of all the nodes of $S_i$. Then, for any $1 \leq i \leq d - 1$, $h_i$ must be connected in $G'_p$ to both $h_{i+1}$ and $h_{i-1}$. Moreover, $h_0$ must be connected to $h_1$, and $h_{d-1}$ must be connected to $h_d$. Hence, this induces a path starting at $h_0$, and ending at $h_d$, with edges $(h_i, h_{i+1})$ for any $0 \leq i \leq d - 1$. But we also want this labelling to be no-hole, hence this path must be hamiltonian. In other words, if we are able to find a hamiltonian path in $G'_p$, then there exists a homomorphism of $H_d$ into $G'_p$. Clearly, since we have $d + 1$ sets $S_i$, and since each one has a unique image in $G'_p$, we must have $d + 1 \geq 2p$, that is $d \geq 2p - 1$.
Finally we need to show that $G'_p$ contains a hamiltonian path ; it is as follows: $p, 0, p+1, 1, \ldots, i, p+i, i+1, p+i+1, \ldots, p-2, 2p-2, p-1, 2p-1$ (cf. Figure 2(left)). Let $v_j$ be any node of set $S_j$, $0 \leq j \leq d$. If $j = 2i$, we set $h(v_{2i}) = p + i$ for every $0 \leq i \leq p - 1$ and if $j = 2i + 1$, we set $h(v_{2i+1}) = i$ for every $0 \leq i \leq p - 1$. Finally, for any $j \geq 2p$, if $j$ is of the form $2p + 2i$, we set $h(v_j) = 2p - 1$, and if $j$ is of the form $2p + 2i + 1$, we set $h(v_j) = 0$.

We can show that the above result can be extended for any $d \geq p + 1$. This is obtained using the same kind of argument (that is, homomorphism into $G'_p$), but with a better mapping of the nodes (cf. Figure 2(right)).

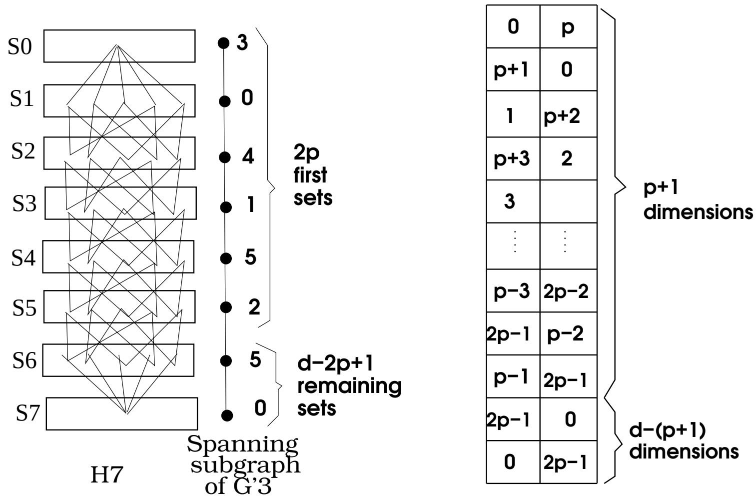


**Fig. 2.** (left) Homomorphism of $H_d$ into $G'_p$, with $d = 7$ and $p = 3$ ; (right) Another homomorphism of $H_d$ into $G'_p$, where nodes are represented by squares

The same goes to extending the result for any $d \geq \frac{p+4}{2}$, hence proving the proposition. Here again, the proof technique is the same as for the two previous cases. Roughly speaking, we consider $H_d$ as 4 copies of $H_{d-2}$, connected between them by 2 perfect matchings. □

**Observation 2** *For any* $p \geq 2$*, if* $\nu_0^p(H_d) = 2p - 1$*, then* $p \leq 2^{d-1} - d + 1$*.*

*Proof.* Suppose that a no-hole $L(p, 0)$ labelling of $H_d$ exists, with $\nu_0^p(H_d) = 2p - 1$. Since $H_d$ has $2^d$ vertices, we must have $2^d \geq 2p$. Since $p \geq 2$, labels $0$,$p - 1$,$p$ and $2p - 1$ are pairwise distinct. Moreover, by definition, there must exist a vertex labelled $p$, whose $d$ neighbours must then be labelled 0. The same goes for any vertex labelled $p - 1$, whose $d$ neighbours must then be labelled $2p-1$. Hence, those four pairwise different labels are present on at least $2(d+1)$ vertices, leaving $2p - 4$ labels to be present on at most $2^d - 2(d + 1)$ vertices. Thus, we must have $2p - 4 \leq 2^d - 2(d + 1)$, which gives the result.

**Proposition 4.** *If* $\nu_0^p(H_{d_0}) = 2p - 1$ *for a given dimension* $d_0$*, then* $\nu_0^p(H_{d'}) = 2p - 1$ *for any* $d' \geq d_0$*.*

*Proof.* Consider $H_{d_0}$, for which we have a no-hole $L(p, 0)$ labelling with $\nu_0^p(H_{d_0}) = 2p - 1$. We will show here a way to obtain a no-hole $L(p, 0)$ labelling of $H_{d_0+1}$ from the labelling of $H_{d_0}$. We recall that $H_{d_0+1}$ is obtained from two copies of $H_{d_0}$ joined by a perfect matching. Now consider a copy of $H_{d_0}$, having a no-hole $L(p, 0)$ labelling with $\nu_0^p(H_{d_0}) = 2p - 1$: necessarily, there must exist a vertex in $H_{d_0}$, say $x$, whose label is $p$. Wlog (since hypercubes are vertex transitive), let the binary coordinates of $x$ be as follows: $x = (0, 0 \ldots 0)$. Since $\nu_0^p(H_{d_0}) = 2p-1$, all the neighbours of $x$ must be labelled 0. Moreover, all the neighbours of $x$ have exactly one bit equal to 1. By the same argument, we can see that all the vertices having 2 bits equal to 1 must be labelled in the range $[p; 2p-1]$, while all the vertices having 3 bits equal to 1 must be labelled in the range $[0; p-1]$. More generally, all the vertices having an even (resp. odd) number of bits equal to 1 are labelled in the range $[p; 2p-1]$ (resp. $[0; p-1]$). Now, take a second copy of $H_{d_0}$, and label each vertex having an even (resp. odd) number of bits equal to 1 with label 0 (resp. $2p-1$). This labelling is an $L(p,0)$ labelling, but it is not no-hole. If we connect the corresponding vertices in both copies of $H_{d_0}$, the labelling we obtain remains $L(p,0)$ (vertices labelled in the range $[p; 2p-1]$ (resp. $[0; p-1]$) in the first copy are connected to vertices labelled 0 (resp. $2p-1$) in the second copy). Moreover, since it is no-hole in the first copy of $H_{d_0}$, it remains no-hole in $H_{d_0+1}$.

**Proposition 5 (2-Dimensional grids $P_n \times P_m$).** *For any $p$ and $n \geq m \geq 1$ we have: $\nu_0^p(P_n \times P_m) = 2p-1$, where*

1. *$n \cdot m - m + 1 \geq 2p$ if $n$ is even and $m$ is odd,*
2. *$n \cdot m - m \geq 2p$ otherwise.*

**Sketch of Proof:** Fill in the $P_n \times P_m$ grid (i.e. $n$ rows and $m$ columns) in the chessboard mode. Like in the chessboard where we have white and black alternating squares, we have in the "white" squares the labels from the range $[p; 2p-1]$ and in the "black" squares the labels $[0; p-1]$. Without loss of generality assume that the left upper square is white. Take the following labelling: put $p$ in the left upper corner and subsequently put in the white squares from left to right and row by row the upper range labels: $p+1, p+2, ..., 2p-1$. In the last row put in all white squares $2p-1$. Further put 0 into all "black" squares of the first row of the grid. Starting with the left most square in the second row of the grid, we subsequently put into the "black" squares lower range labels from $[0; p-1]$. The labelling is no-hole $L(p,0)$ and $2p = m(n-1)+1$ if $m$ is odd and $n$ is even. Otherwise $2p = m(n-1)$. □

Below is an example of 2-dimensional grids $G(5,6)$ and $G(5,5)$.

**Proposition 6 (Consequence of Proposition 5).** *For any $p, q \geq 0$ and $d \geq 4$ we have: $\nu_0^p(H_d) = 2p-1$ when*

1. *$(\lfloor d/2 \rfloor + 1)\lceil d/2 \rceil + 2 \geq 2p$ if $d$ is odd,*
2. *$(d/2+1)^2 - d/2 \geq 2p$ if $d$ is even.*

*Proof.* Proof follows by combining Propositions 3 and 5.

*Remark 1.* Similar results to Propositions 5 and 6 can be obtained for the 3 and higher dimensional grids. The results will appear in the full version of this work. Below is an example of a no-hole labelling of a 3-dimensional grid (see Figure 4).

By direct application of Proposition 1, we get the following result for the 2-dimensional toroidal meshes.

| | | | | |
|---|---|---|---|---|
| 13 | 0 | 14 | 0 | 15 |
| 0 | 16 | 1 | 17 | 2 |
| 18 | 3 | 19 | 4 | 20 |
| 5 | 21 | 6 | 22 | 7 |
| 23 | 8 | 24 | 9 | 25 |
| 10 | 25 | 11 | 25 | 12 |

| | | | | |
|---|---|---|---|---|
| 10 | 0 | 11 | 0 | 12 |
| 0 | 13 | 1 | 14 | 2 |
| 15 | 3 | 16 | 4 | 17 |
| 5 | 18 | 6 | 19 | 7 |
| 19 | 8 | 19 | 9 | 19 |

**Fig. 3.** No-hole $L(p, 0)$ labelling in 2D grids $G(n, m)$: (left) $m = 5, n = 6, p = 13$; (right) $m = n = 5, p = 10$

| | | | |
|---|---|---|---|
| 24 | 0 | 25 | 1 |
| 0 | 26 | 1 | 31 |
| 27 | 2 | 32 | 7 |
| 3 | 33 | 8 | 43 |

| | | | |
|---|---|---|---|
| 0 | 28 | 1 | 34 |
| 29 | 2 | 35 | 7 |
| 3 | 36 | 8 | 44 |
| 37 | 9 | 45 | 17 |

| | | | |
|---|---|---|---|
| 30 | 4 | 38 | 10 |
| 5 | 39 | 11 | 46 |
| 40 | 12 | 47 | 18 |
| 13 | 47 | 19 | 47 |

| | | | |
|---|---|---|---|
| 6 | 41 | 14 | 47 |
| 42 | 15 | 47 | 20 |
| 16 | 47 | 21 | 47 |
| 47 | 22 | 47 | 23 |

**Fig. 4.** A $L(24, 0)$ no-hole labelling in $P_4 \times P_4 \times P_4$ with $\nu_0^{24} = 47$. Each of the 4 blocks represents a 2-Dimensional subgrid $P_4 \times P_4$, in which each square represents a node

**Proposition 7 (Tori).** *For any* $p \geq 1$ *and* $n, m \geq 3$*:*

- $\nu_0^p(C_n \times C_m) \geq 2p - 1$ *if* $n$ *and* $m$ *are both even*
- $\nu_0^p(C_n \times C_m) \geq 2p$ *otherwise*

| | | | |
|---|---|---|---|
| 7 | 0 | 7 | 3 |
| 0 | 4 | 0 | 7 |
| 6 | 0 | 6 | 2 |
| 1 | 5 | 0 | 7 |

**Fig. 5.** $L(p, 0)$ no-hole labelling in 2D-tori; an example for $C_4 \times C_4$ where $p = 4$

## 3 $L(p, q)$ Labellings of Hypercubes

We conclude this paper by giving new results concerning the (general, i.e. not necessarily no-hole) $L(p, q)$ labelling of $H_d$. Some results on this topic have been given in [7]. However, it is possible to improve them. This is the purpose of Proposition 8 below.

**Proposition 8.** *For any $p, q, \geq 0$ and $d \geq 1$:*

1. $\lambda_0^p(H_d) = p$
2. $(d-1)q \leq \lambda_q^0(H_d) \leq (2d-3)q$
3. $\lambda_q^p(H_d) \leq 2p + (2d-2)q - 1$.

*Proof.* (1) For any $d \geq 1$, $H_d$ has at least two vertices. Consider a vertex $v$ of $H_d$ that is assigned label 0 in a $L(p,0)$ labelling of $H_d$. Such a vertex must exist, otherwise every label could be decreased by at least 1, leading to a better solution. $v$ has at least one neighbour $w$, whose label must then be greater than or equal to $p$. Hence, $\nu_0^p(H_d) \geq p$. We can show that $\nu_0^p(H_d) \leq p$ by noticing that $H_d$ is bipartite. Thus, if all the vertices of the first (resp. second) partition of $H_d$ are labelled 0 (resp. $p$), the labelling we obtain is $L(p,0)$.

(2) Take any vertex $u$ of $H_d$. It has $d$ neighbors, all of them lying at distance 2 from each other. Hence, those $d$ vertices must be assigned labels that differ by at least $q$. Since labels can begin at 0, it follows that the greatest label is greater than or equal to $(d-1)q$, showing that $\lambda_q^O(H_d) \geq (d-1)q$. The upper bound is obtained by the following labelling: suppose each node $v = (x_1, x_2 \dots x_d)$ is defined by its (binary) coordinates in each of the $d$ dimensions of $H_d$ ; for any vertex $v = (x_1, x_2 \dots x_d)$ of $G_d$, we define $c(v) = (\sum_{k=1}^{d-1} kqx_k) \bmod (2d-2)q$. Since $p = 0$, we only need to consider two vertices $u$ and $v$ lying at distance 2 in $G_d$, thus differing on two coordinates, say $x_i$ and $x_j$, $1 \leq i \neq j \leq d$. We will consider two cases here: (a) $j = d$ and (b) $j \neq d$. In case (a), we have $|c(u) - c(v)| = iq \bmod (2d-2)q$, and since $1 \leq i \leq d-1$, $|c(u) - c(v)| \geq q$. In case (b), we either have $|c(u) - c(v)| = (i+j)q \bmod (2d-2)q$ or $|c(u) - c(v)| = (j-i)q \bmod (2d-2)q$, but since $1 \leq i \leq d-2$ and $i+1 \leq j \leq d-1$, $|c(u) - c(v)| \geq q$.

(3) Suppose each node $v = (x_1, x_2 \dots x_d)$ is defined by its (binary) coordinates in each of the $d$ dimensions of $H_d$. Every node $v = (x_1, x_2 \dots x_d)$ is then assigned label $c(v) = \sum_{i=1}^{d}(p + (i-1)q)x_i \bmod (2p + (2d-2)q)$. Take two neighbors $u$ and $v$ in $H_d$, which thus differ on exactly one coordinate, say in $x_j$. Thus $|c(v) - c(u)| = p + (j-1)q \bmod (2p + (2d-2)q)$, that is $|c(v) - c(u)| = p + (j-1)q$. Since $1 \leq j \leq d$, we have that $|c(v) - c(u)| \geq p$. Now consider two nodes differing on two coordinates $i$ and $j$, where, wlog, $i < j$ (since we are in $H_d$, necessarily $i \neq j$). We have two cases here: (a) $x_i = x_j$ or (b) $x_i \neq x_j$. In case (a), we obtain that $|c(v) - c(u)| = 2p + (i+j-2)q$, which is clearly greater than or equal to $q$. In case (b), $|c(v) - c(u)| = (j-i)q$, which is also greater than or equal to $q$ since $j \neq i$.

*Remark 2.*

- Concerning the $L(0,q)$ labelling of $H_d$, we can show that $\lambda_q^0(H_3) = \lambda_q^0(H_4) = 3q$ ;

– We note that Proposition 8(3), when applied to the case $p = 2$ and $q = 1$, gives $\lambda_q^p(H_d) \leq 2d + 1$, a value which coincides with the lower bound proved in [8].

## 4 Conclusion

In this paper, we have mainly considered the no-hole $L(p, 0)$ labelling in different topologies, such as cycles, hypercubes and 2-dimensional tori. We also gave some bounds for the (not necessarily no-hole) $L(p, q)$ labelling of hypercubes, that improve the ones from [7].
Concerning no-hole $L(p, 0)$ labellings, we have mainly based our study on the cases for which there exists a no-hole $L(p, 0)$ labelling having the minimum number of labels (that is, minimum as stated in Proposition 1). Indeed, depending on the respective values of $n$ (number of nodes of the considered network) and $p$, such a no-hole $L(p, 0)$ labelling might not exist. Also, as remarked above, our work concerning 2-dimensional grids can be extended to any $d$-dimensional grids, $d \geq 3$ (this will appear in the full version of this work). We also note that a natural extension of this work is to study the no-hole $L(p, q)$ labelling of graphs, for any $p, q$.

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