

No-Hole $L(p, 0)$ Labelling of Cycles, Grids and Hypercubes*

Guillaume Fertin¹, André Raspaud², and Ondrej Sýkora³

¹ LINA, FRE CNRS 2729

Université de Nantes, 2 rue de la Houssinière
BP 92208 44322 Nantes Cedex 3 - FRANCE
`fertin@lina.univ-nantes.fr`

² LaBRI U.M.R. 5800, Université Bordeaux 1
351 Cours de la Libération - F33405 Talence Cedex
`raspaud@labri.fr`

³ Department of Computer Science, Loughborough University
LE11 3TU - The United Kingdom `O.Sykora@lboro.ac.uk`

Abstract. In this paper, we address a particular case of the general problem of λ labellings, concerning frequency assignment for telecommunication networks. In this model, stations within a given radius r must use frequencies that differ at least by a value p , while stations that are within a larger radius $r' > r$ must use frequencies that differ by at least another value q . The aim is to minimize the span of frequencies used in the network. This can be modelled by a graph labelling problem, called the $L(p, q)$ labelling, where one wants to label vertices of the graph G modelling the network by integers in the range $[0; M]$, while minimizing the value of M . M is then called the λ number of G , and is denoted by $\lambda_q^p(G)$.

Another parameter that sometimes needs to be optimized is the fact that all the possible frequencies (i.e., all the possible values in the span) are used. In this paper, we focus on this problem. More precisely, we want that: (1) all the frequencies are used and (2) condition (1) being satisfied, the span must be minimum. We call this the *no-hole* $L(p, q)$ labelling problem for G . Let $[0; M']$ be this new span and call the ν number of G the value M' , and denote it by $\nu_q^p(G)$.

In this paper, we study a special case of no-hole $L(p, q)$ labelling, namely where $q = 0$. We also focus on some specific topologies: cycles, hypercubes, 2-dimensional grids and 2-dimensional tori. For each of the mentioned topologies cited above, we give bounds on the ν_0^p number and show optimality in some cases. The paper is concluded by giving new results concerning the (general, i.e. not necessarily no-hole) $L(p, q)$ labelling of hypercubes.

* This work was done while the two first authors were visiting the University of Loughborough, and was supported in part by the EPSRC grant GR/R37395/01 and by VEGA grant No. 2/3164/23.

1 Introduction

In this paper, we study the *frequency assignment problem*, that arises in wireless communication systems. We are interested here in minimizing the number of frequencies used in the framework where radio transmitters that are geographically close may interfere if they are assigned close frequencies. This problem has originally been introduced in [12] and later developed in [9], where it has been shown to be equivalent to a graph labelling problem, in which the nodes represent the transmitters, and any edge joins two transmitters that are sufficiently close to potentially interfere. The aim here is to label the nodes of the graph in such a way that:

- any two neighbours (transmitters that are very close) are assigned labels (frequencies) that differ by a parameter at least p ;
- any two vertices at distance 2 (transmitters that are close) are assigned labels (frequencies) that differ by a parameter at least q ;
- the greatest value for the labels is minimized.

It has been proved that under this model, we could assume the labels to be integers, starting at 0 [8]. In that case, the minimum range of frequencies that is necessary to assign to the vertices of a graph G is denoted $\lambda_q^p(G)$, and the problem itself is usually called the $L(p, q)$ labelling problem. The frequency assignment problem has been studied in many different specific topologies [8, 14, 17, 1, 3, 5, 13, 2, 15, 16]. The case $p = 2$ and $q = 1$ is the most widely studied (see for instance [6, 11, 10, 4]). Some variants of the model also exist, such as the following generalization where one gives k constraints on the k first distances (any two vertices at distance $1 \leq i \leq k$ in G must be assigned labels differing by at least δ_i). One of the issues also considered in the frequency assignment problem is the *no-hole labelling*, where one wants to use all the frequencies in the span. More precisely, we want that: (1) all the frequencies are used and (2) condition (1) being satisfied, the span must be minimum. Let $[0; M']$ be the span of frequencies that we obtain. We then call the ν number of G the value M' , and we denote it by $\nu_q^p(G)$. We note that depending on the values of p , q and on the considered graph G , a no-hole labelling might not exist. In that case, we let $\nu_q^p(G) = \infty$. Hence, we clearly have $\nu_q^p(G) \geq \lambda_q^p(G)$ for any p , q and G .

In this paper, we study a special case of no-hole $L(p, q)$ labelling, namely where $q = 0$. We also focus on some specific topologies: cycles, d -dimensional hypercubes, 2-dimensional grids and 2-dimensional tori. For each of the mentioned topologies cited above, we give bounds on the ν_0^p number for any value of $p \geq 1$ and $d \geq 1$, and show optimality in some cases. We conclude the paper with new results concerning the (general, i.e. not necessarily no-hole) $L(p, q)$ labelling of the d -dimensional hypercube, H_d .

2 No-Hole $L(p, 0)$ labellings

Proposition 1 (General graphs). *For any $p \geq 1$ and any connected graph G , if a no-hole labelling of G exists, then:*

- $\nu_0^p(G) \geq 2p - 1$ if G is bipartite.
- $\nu_0^p(G) \geq 2p$ if G is not bipartite.

Proof. In any graph such that a no-hole $L(p, 0)$ labelling exists, there must, by definition, exist at least one vertex of label 0. Let u be this vertex. Then all the neighbours of u must be labelled at least p . Since the labelling is no-hole, all the labels in the range $[0; p]$ must be used. This is true, in particular, for label $p - 1$. Let v be a vertex whose label is $p - 1$. Then v has all its neighbours labelled at least $2p - 1$.

Now, let G be non bipartite, and suppose that $\nu_0^p(G) \leq 2p - 1$. G has at least one odd cycle. Consider the vertices on this cycle, and let $i \geq 0$ be the minimum label among them, assigned to vertex v . If $i \geq p$, then the neighbours of v , not being of minimum label, must be assigned a label at least $2p$, a contradiction. Hence, $i \in [0; p - 1]$. In that case, the two neighbours of v on the cycle, say w_1 and w_2 , are assigned labels at least $p + i$, that is in the range $[p; 2p - 1]$. But the neighbours of w_1 and w_2 on the cycle must be assigned labels in the range $[0; p - 1]$, etc. If we repeat this argument, we see that, when we will close the cycle, since it is odd, we will end up with a vertex z whose two neighbours, say x and y , are such that x is assigned a label in the range $[0; p - 1]$, while y is assigned a label in the range $[p; 2p - 1]$. As $\nu_0^p(G) \leq 2p - 1$, there is no possibility to label z in an $L(p, 0)$ fashion, a contradiction.

Observation 1 For any graph G of order n that admits a no-hole labelling, $n \geq \nu_0^p(G) + 1$

Proof. Suppose a no-hole labelling for G exists. In order to be able to assign the vertices of G all labels in the range $[0; \nu_0^p(G)]$, we must have $n \geq \nu_0^p(G) + 1$.

Proposition 2 (Cycles). For any $p \geq 1$ and any graph G :

- $\nu_0^p(C_n) = 2p$ for any odd $n \geq 2p + 1$
- $\nu_0^p(C_n) = 2p - 1$ for any even $n \geq 2p + 2$

Proof. First, suppose that n is even. According to Proposition 1, we know that, if a no-hole $L(p, 0)$ labelling for C_n exists, then $\nu_0^p(C_n) \geq 2p - 1$. By Observation 1, we must have $n \geq 2p$. However, let v be the vertex which is assigned label p . In that case, both its neighbours must be assigned label 0, because only labels in the range $[0; 2p - 1]$ are allowed, and the gap between two neighbours is at least p . Hence, $n \geq 2p + 1$; but since n is even, we have $n \geq 2p + 2$. Suppose the vertices of C_n are numbered clockwise from 1 to n . We give the following labelling function c on C_n : (a) for any vertex v numbered $2i + 1$ ($0 \leq i \leq p - 1$), $c(v) = p + i$; (b) for any vertex v numbered $2i$ ($1 \leq i \leq p$), $c(v) = i - 1$; (c) for any vertex v numbered $2i + 1$, $i \geq p$, $c(v) = 2p - 1$; (d) for any vertex v numbered $2i$ ($i \geq p + 1$), $c(v) = 0$ (cf. for instance Figure 1 (right), where $n = 2p + 2$). Now, we show that this assignment is an $L(p, 0)$ no-hole labelling of C_n , for any even $n \geq 2p + 2$. First, consider any two neighbouring vertices j and $j + 1$, $1 \leq j \leq 2p - 1$. If j is even $j = 2i$, then $c(j) = i - 1$, while $c(j + 1) = p + i$,

thus the gap of at least p is satisfied. If j is odd $j = 2i + 1$, then $c(j) = p + i$, while $c(j + 1) = i$, which also satisfies the $L(p, 0)$ condition. Now consider two neighbours j and $j + 1$, with $2p + 1 \leq j \leq n - 1$. If j is even, then $c(j) = 0$ while $c(j + 1) = 2p - 1$, and if j is odd, then $c(j) = 2p - 1$ while $c(j + 1) = 0$. There still remain two cases to consider: (1) Vertices 1 and n and (2) vertices $2p$ and $2p + 1$. However, in case (1) we have $c(1) = p$ and $c(n) = 0$, while in case (2) we have $c(2p) = p - 1$ and $c(2p + 1) = 2p - 1$.

Consequently, the $L(p, 0)$ condition is satisfied. There is no hole as from the definition of the labelling, one can see that all labels are used on vertices 0 to $2p$: odd vertices $2i + 1$, $0 \leq i \leq p - 1$ have labels from p to $2p - 1$, while even vertices $2i$, $1 \leq i \leq p$ are assigned labels from 0 to $p - 1$.

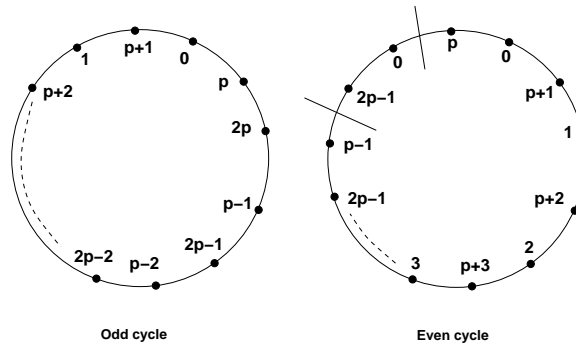


Fig. 1. $L(p, 0)$ no-hole labelling in cycles

Suppose that n is odd. By Proposition 1, we know that, if a no-hole $L(p, 0)$ labelling for C_n exists, then $\nu_0^p(C_n) \geq 2p$. Now, clearly, if $n \leq 2p - 1$, then there is not enough vertices to use all the labels. Thus, we must have $n \geq 2p + 1$ in order that the no-hole labelling exists. Assume that $n \geq 2p + 1$, and the vertices of C_n are numbered clockwise from 1 to n . We define the following labelling function c to C_n : (a) for any vertex v numbered $2i + 1$ ($0 \leq i \leq p$), $c(v) = p - i$; (b) for any vertex v numbered $2i$ ($1 \leq i \leq p$), $c(v) = 2p - i + 1$; (c) for any vertex v numbered $2i + 1$, $i \geq p + 1$, $c(v) = 0$; (d) for any vertex v numbered $2i$ ($i \geq p + 1$), $c(v) = p$ (cf. for instance Figure 1 (left), where $n = 2p + 1$).

The labelling is an $L(p, 0)$ no-hole labelling of C_n , for any odd $n \geq 2p + 1$. First, consider any two neighbouring vertices j and $j + 1$, $1 \leq j \leq 2p$. If j is even $j = 2i$, then $c(j) = 2p - i + 1$, while $c(j + 1) = p - i$, thus the gap of at least p is satisfied. If j is odd $j = 2i + 1$, then $c(j) = p - i$, while $c(j + 1) = 2p - i$, which also fulfills the $L(p, 0)$ condition. Now consider two neighbours j and $j + 1$, with $2p + 2 \leq j \leq n - 1$. If j is even, then $c(j) = p$ while $c(j + 1) = 0$, and if j is odd, then $c(j) = 0$ while $c(j + 1) = p$. Now there remains some cases to consider: (1) $j = 2p + 1$ and $j + 1 = 2p + 2$ and (2) $j = n$ and $j + 1 = 0$ (that is, we

“close” the cycle). But in both cases, we have $c(j) = 0$ and $c(j + 1) = p$. Thus, altogether, the $L(p, 0)$ condition is satisfied. Now, by definition of the labelling, we can see that all the labels are used on vertices 0 to $2p$: vertices of the form $2i + 1$, $0 \leq i \leq p$ are assigned labels from 0 to p , while vertices of the form $2i$, $1 \leq i \leq p$ are assigned labels from $p + 1$ to $2p$.

Proposition 3 (Hypercubes). *For any d -dimensional hypercube H_d such that $d \geq \frac{p+4}{2}$, $\nu_0^p(H_d) = 2p - 1$.*

Sketch of Proof: By Proposition 1, $\nu_0^p(H_d) \geq 2p - 1$. We will first show that $\nu_0^p(H_d) \leq 2p - 1$ (thus, proving the equality) for any $d \geq 2p - 1$; then, we will show that this result can be extended to any $d \geq \frac{p+4}{2}$. Suppose $d \geq 2p - 1$. The fact that $\nu_0^p(H_d) \leq 2p - 1$ is proved by homomorphism into the following graph G'_p : (a) the nodes of G'_p are the integers between 0 and $2p - 1$ and (b) there is an edge between u and v in G'_p iff $|u - v| \geq p$. Clearly, G'_p represents the constraints on the $L(p, 0)$ labelling, in the sense that any edge (u, v) of G'_p indicate that labels u and v can be assigned to neighboring nodes in H_d . We want to find a homomorphism \mathcal{H} from H_d to G'_p , i.e. we want to find a mapping from $V(H_d)$ to $V(G')$, where every node v has an image $h(v)$ such that any edge (u, v) in H_d corresponds to an edge $(h(u), h(v))$ in G'_p . If we can do this, then we can find a labelling (more precisely, $c(v) = h(v)$ for any node v) that satisfies the $L(p, 0)$ constraints. Furthermore as this labelling has to be no-hole, we also need that every node of G'_p is an image of at least one node of H_d . Let us partition the nodes of H_d into $d + 1$ sets: for any $0 \leq i \leq d$, the set S_i corresponds to the nodes having i bits equal to 0 in its binary coordinates. By definition of the hypercube, for every $0 \leq i \leq d$, S_i is a stable set. In other words, all edges appear between different S_i s. More precisely, all the edges of H_d appear between an S_i and an S_{i+1} . Let us define the homomorphism so that all nodes belonging to the same S_i have the same image by \mathcal{H} . Let h_i be the image by \mathcal{H} of all the nodes of S_i . Then, for any $1 \leq i \leq d - 1$, h_i must be connected in G'_p to both h_{i+1} and h_{i-1} . Moreover, h_0 must be connected to h_1 , and h_{d-1} must be connected to h_d . Hence, this induces a path starting at h_0 , and ending at h_d , with edges (h_i, h_{i+1}) for any $0 \leq i \leq d - 1$. But we also want this labelling to be no-hole, hence this path must be hamiltonian. In other words, if we are able to find a hamiltonian path in G'_p , then there exists a homomorphism of H_d into G'_p . Clearly, since we have $d + 1$ sets S_i , and since each one has a unique image in G'_p , we must have $d + 1 \geq 2p$, that is $d \geq 2p - 1$.

Finally we need to show that G'_p contains a hamiltonian path; it is as follows: $p, 0, p+1, 1, \dots, i, p+i, i+1, p+i+1, \dots, p-2, 2p-2, p-1, 2p-1$ (cf. Figure 2(left)). Let v_j be any node of set S_j , $0 \leq j \leq d$. If $j = 2i$, we set $h(v_{2i}) = p + i$ for every $0 \leq i \leq p - 1$ and if $j = 2i + 1$, we set $h(v_{2i+1}) = i$ for every $0 \leq i \leq p - 1$. Finally, for any $j \geq 2p$, if j is of the form $2p + 2i$, we set $h(v_j) = 2p - 1$, and if j is of the form $2p + 2i + 1$, we set $h(v_j) = 0$.

We can show that the above result can be extended for any $d \geq p + 1$. This is obtained using the same kind of argument (that is, homomorphism into G'_p), but with a better mapping of the nodes (cf. Figure 2(right)).

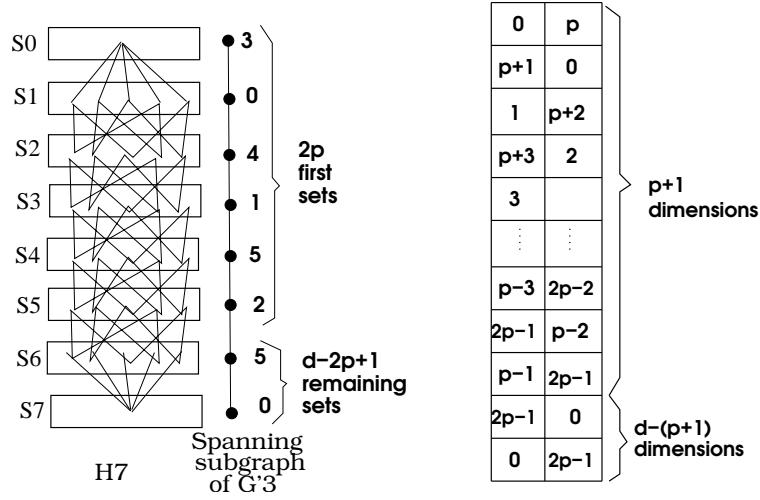


Fig. 2. (left) Homomorphism of H_d into G'_p , with $d = 7$ and $p = 3$; (right) Another homomorphism of H_d into G'_p , where nodes are represented by squares

The same goes to extending the result for any $d \geq \frac{p+4}{2}$, hence proving the proposition. Here again, the proof technique is the same as for the two previous cases. Roughly speaking, we consider H_d as 4 copies of H_{d-2} , connected between them by 2 perfect matchings. \square

Observation 2 For any $p \geq 2$, if $\nu_0^p(H_d) = 2p - 1$, then $p \leq 2^{d-1} - d + 1$.

Proof. Suppose that a no-hole $L(p, 0)$ labelling of H_d exists, with $\nu_0^p(H_d) = 2p - 1$. Since H_d has 2^d vertices, we must have $2^d \geq 2p$. Since $p \geq 2$, labels $0, p - 1, p$ and $2p - 1$ are pairwise distinct. Moreover, by definition, there must exist a vertex labelled p , whose d neighbours must then be labelled 0 . The same goes for any vertex labelled $p - 1$, whose d neighbours must then be labelled $2p - 1$. Hence, those four pairwise different labels are present on at least $2(d + 1)$ vertices, leaving $2p - 4$ labels to be present on at most $2^d - 2(d + 1)$ vertices. Thus, we must have $2p - 4 \leq 2^d - 2(d + 1)$, which gives the result.

Proposition 4. If $\nu_0^p(H_{d_0}) = 2p - 1$ for a given dimension d_0 , then $\nu_0^p(H_{d'}) = 2p - 1$ for any $d' \geq d_0$.

Proof. Consider H_{d_0} , for which we have a no-hole $L(p, 0)$ labelling with $\nu_0^p(H_{d_0}) = 2p - 1$. We will show here a way to obtain a no-hole $L(p, 0)$ labelling of H_{d_0+1} from the labelling of H_{d_0} . We recall that H_{d_0+1} is obtained from two copies of H_{d_0} joined by a perfect matching. Now consider a copy of H_{d_0} , having a no-hole $L(p, 0)$ labelling with $\nu_0^p(H_{d_0}) = 2p - 1$: necessarily, there must exist a vertex in H_{d_0} , say x , whose label is p . Wlog (since hypercubes are vertex transitive), let the binary coordinates of x be as follows: $x = (0, 0 \dots 0)$. Since $\nu_0^p(H_{d_0}) = 2p - 1$,

all the neighbours of x must be labelled 0. Moreover, all the neighbours of x have exactly one bit equal to 1. By the same argument, we can see that all the vertices having 2 bits equal to 1 must be labelled in the range $[p; 2p - 1]$, while all the vertices having 3 bits equal to 1 must be labelled in the range $[0; p - 1]$. More generally, all the vertices having an even (resp. odd) number of bits equal to 1 are labelled in the range $[p; 2p - 1]$ (resp. $[0; p - 1]$). Now, take a second copy of H_{d_0} , and label each vertex having an even (resp. odd) number of bits equal to 1 with label 0 (resp. $2p - 1$). This labelling is an $L(p, 0)$ labelling, but it is not no-hole. If we connect the corresponding vertices in both copies of H_{d_0} , the labelling we obtain remains $L(p, 0)$ (vertices labelled in the range $[p; 2p - 1]$ (resp. $[0; p - 1]$) in the first copy are connected to vertices labelled 0 (resp. $2p - 1$) in the second copy). Moreover, since it is no-hole in the first copy of H_{d_0} , it remains no-hole in H_{d_0+1} .

Proposition 5 (2-Dimensional grids $P_n \times P_m$). *For any p and $n \geq m \geq 1$ we have: $\nu_0^p(P_n \times P_m) = 2p - 1$, where*

1. $n \cdot m - m + 1 \geq 2p$ if n is even and m is odd,
2. $n \cdot m - m \geq 2p$ otherwise.

Sketch of Proof: Fill in the $P_n \times P_m$ grid (i.e. n rows and m columns) in the chessboard mode. Like in the chessboard where we have white and black alternating squares, we have in the “white” squares the labels from the range $[p; 2p - 1]$ and in the “black” squares the labels $[0; p - 1]$. Without loss of generality assume that the left upper square is white. Take the following labelling: put p in the left upper corner and subsequently put in the white squares from left to right and row by row the upper range labels: $p + 1, p + 2, \dots, 2p - 1$. In the last row put in all white squares $2p - 1$. Further put 0 into all “black” squares of the first row of the grid. Starting with the left most square in the second row of the grid, we subsequently put into the “black” squares lower range labels from $[0; p - 1]$. The labelling is no-hole $L(p, 0)$ and $2p = m(n - 1) + 1$ if m is odd and n is even. Otherwise $2p = m(n - 1)$. \square

Below is an example of 2-dimensional grids $G(5, 6)$ and $G(5, 5)$.

Proposition 6 (Consequence of Proposition 5). *For any $p, q \geq 0$ and $d \geq 4$ we have: $\nu_0^p(H_d) = 2p - 1$ when*

1. $(\lfloor d/2 \rfloor + 1)\lceil d/2 \rceil + 2 \geq 2p$ if d is odd,
2. $(d/2 + 1)^2 - d/2 \geq 2p$ if d is even.

Proof. Proof follows by combining Propositions 3 and 5.

Remark 1. Similar results to Propositions 5 and 6 can be obtained for the 3 and higher dimensional grids. The results will appear in the full version of this work. Below is an example of a no-hole labelling of a 3-dimensional grid (see Figure 4).

By direct application of Proposition 1, we get the following result for the 2-dimensional toroidal meshes.

13	0	14	0	15
0	16	1	17	2
18	3	19	4	20
5	21	6	22	7
23	8	24	9	25
10	25	11	25	12

10	0	11	0	12
0	13	1	14	2
15	3	16	4	17
5	18	6	19	7
19	8	19	9	19

Fig. 3. No-hole $L(p, 0)$ labelling in 2D grids $G(n, m)$: (left) $m = 5, n = 6, p = 13$; (right) $m = n = 5, p = 10$

24	0	25	1
0	26	1	31
27	2	32	7
3	33	8	43

0	28	1	34
29	2	35	7
3	36	8	44
37	9	45	17

30	4	38	10
5	39	11	46
40	12	47	18
13	47	19	47

6	41	14	47
42	15	47	20
16	47	21	47
47	22	47	23

Fig. 4. A $L(24, 0)$ no-hole labelling in $P_4 \times P_4 \times P_4$ with $\nu_0^{24} = 47$. Each of the 4 blocks represents a 2-Dimensional subgrid $P_4 \times P_4$, in which each square represents a node

Proposition 7 (Tori). For any $p \geq 1$ and $n, m \geq 3$:

- $\nu_0^p(C_n \times C_m) \geq 2p - 1$ if n and m are both even
- $\nu_0^p(C_n \times C_m) \geq 2p$ otherwise

7	0	7	3
0	4	0	7
6	0	6	2
1	5	0	7

Fig. 5. $L(p, 0)$ no-hole labelling in 2D-tori; an example for $C_4 \times C_4$ where $p = 4$

3 $L(p, q)$ Labellings of Hypercubes

We conclude this paper by giving new results concerning the (general, i.e. not necessarily no-hole) $L(p, q)$ labelling of H_d . Some results on this topic have been

given in [7]. However, it is possible to improve them. This is the purpose of Proposition 8 below.

Proposition 8. *For any $p, q, \geq 0$ and $d \geq 1$:*

1. $\lambda_0^p(H_d) = p$
2. $(d-1)q \leq \lambda_q^0(H_d) \leq (2d-3)q$
3. $\lambda_q^p(H_d) \leq 2p + (2d-2)q - 1$.

Proof. (1) For any $d \geq 1$, H_d has at least two vertices. Consider a vertex v of H_d that is assigned label 0 in a $L(p, 0)$ labelling of H_d . Such a vertex must exist, otherwise every label could be decreased by at least 1, leading to a better solution. v has at least one neighbour w , whose label must then be greater than or equal to p . Hence, $\nu_0^p(H_d) \geq p$. We can show that $\nu_0^p(H_d) \leq p$ by noticing that H_d is bipartite. Thus, if all the vertices of the first (resp. second) partition of H_d are labelled 0 (resp. p), the labelling we obtain is $L(p, 0)$.

(2) Take any vertex u of H_d . It has d neighbors, all of them lying at distance 2 from each other. Hence, those d vertices must be assigned labels that differ by at least q . Since labels can begin at 0, it follows that the greatest label is greater than or equal to $(d-1)q$, showing that $\lambda_q^0(H_d) \geq (d-1)q$. The upper bound is obtained by the following labelling: suppose each node $v = (x_1, x_2 \dots x_d)$ is defined by its (binary) coordinates in each of the d dimensions of H_d ; for any vertex $v = (x_1, x_2 \dots x_d)$ of G_d , we define $c(v) = (\sum_{k=1}^{d-1} kqx_k) \bmod (2d-2)q$. Since $p = 0$, we only need to consider two vertices u and v lying at distance 2 in G_d , thus differing on two coordinates, say x_i and x_j , $1 \leq i \neq j \leq d$. We will consider two cases here: (a) $j = d$ and (b) $j \neq d$. In case (a), we have $|c(u) - c(v)| = iq \bmod (2d-2)q$, and since $1 \leq i \leq d-1$, $|c(u) - c(v)| \geq q$. In case (b), we either have $|c(u) - c(v)| = (i+j)q \bmod (2d-2)q$ or $|c(u) - c(v)| = (j-i)q \bmod (2d-2)q$, but since $1 \leq i \leq d-2$ and $i+1 \leq j \leq d-1$, $|c(u) - c(v)| \geq q$.

(3) Suppose each node $v = (x_1, x_2 \dots x_d)$ is defined by its (binary) coordinates in each of the d dimensions of H_d . Every node $v = (x_1, x_2 \dots x_d)$ is then assigned label $c(v) = \sum_{i=1}^d (p + (i-1)q)x_i \bmod (2p + (2d-2)q)$. Take two neighbors u and v in H_d , which thus differ on exactly one coordinate, say in x_j . Thus $|c(v) - c(u)| = p + (j-1)q \bmod (2p + (2d-2)q)$, that is $|c(v) - c(u)| = p + (j-1)q$. Since $1 \leq j \leq d$, we have that $|c(v) - c(u)| \geq p$. Now consider two nodes differing on two coordinates i and j , where, wlog, $i < j$ (since we are in H_d , necessarily $i \neq j$). We have two cases here: (a) $x_i = x_j$ or (b) $x_i \neq x_j$. In case (a), we obtain that $|c(v) - c(u)| = 2p + (i+j-2)q$, which is clearly greater than or equal to q . In case (b), $|c(v) - c(u)| = (j-i)q$, which is also greater than or equal to q since $j \neq i$.

Remark 2.

- Concerning the $L(0, q)$ labelling of H_d , we can show that $\lambda_q^0(H_3) = \lambda_q^0(H_4) = 3q$;

- We note that Proposition 8(3), when applied to the case $p = 2$ and $q = 1$, gives $\lambda_q^p(H_d) \leq 2d + 1$, a value which coincides with the lower bound proved in [8].

4 Conclusion

In this paper, we have mainly considered the no-hole $L(p, 0)$ labelling in different topologies, such as cycles, hypercubes and 2-dimensional tori. We also gave some bounds for the (not necessarily no-hole) $L(p, q)$ labelling of hypercubes, that improve the ones from [7].

Concerning no-hole $L(p, 0)$ labellings, we have mainly based our study on the cases for which there exists a no-hole $L(p, 0)$ labelling having the minimum number of labels (that is, minimum as stated in Proposition 1). Indeed, depending on the respective values of n (number of nodes of the considered network) and p , such a no-hole $L(p, 0)$ labelling might not exist. Also, as remarked above, our work concerning 2-dimensional grids can be extended to any d -dimensional grids, $d \geq 3$ (this will appear in the full version of this work). We also note that a natural extension of this work is to study the no-hole $L(p, q)$ labelling of graphs, for any p, q .

References

- [1] A.A Bertossi, C.M. Pinotti, and R.B. Tan. Efficient use of radio spectrum in wireless networks with channel separation between close stations. In *Proc. DIAL M for Mobility 2000 (Fourth International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications)*, 2000.
- [2] A.A Bertossi, C.M. Pinotti, and R.B. Tan. Channel assignment with separation for special classes of wireless networks : Grids and rings. In *Proc. IPDPS'02 (International Parallel and Distributed Processing Symposium)*, pages 28–33. IEEE Computer Society, 2002.
- [3] H.L. Bodlaender, T. Kloks, R.B. Tan, and J. van Leeuwen. λ -coloring of graphs. In *Proc. STACS 2000 : 17th Annual Symposium on Theoretical Aspect of Computer Science*, volume 1770, pages 395–406. Lecture Notes Computer Science, Springer-Verlag Berlin, 2000.
- [4] T. Calamoneri and R. Petreschi. $L(2, 1)$ -labeling of planar graphs. In *Proc. DIAL M for Mobility 2001 (Fifth International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications)*, pages 28–33, 2001.
- [5] G.J. Chang, W.-T. Ke, D. Kuo, D.D.-F. Liu, and R.K. Yeh. On $L(d, 1)$ -labelings of graphs. *Discrete Mathematics*, 220:57–66, 2002.
- [6] G.J. Chang and D. Kuo. The $L(2, 1)$ -labeling on graphs. *SIAM J. Discrete Math.*, 9:309–316, 1996.
- [7] G. Fertin and R. Raspaud. $L(p, q)$ labeling of d -dimensional grids. *Discrete Mathematics*, 2003. Submitted.
- [8] J.R. Griggs and R.K. Yeh. Labeling graphs with a condition at distance two. *SIAM J. Discrete Math.*, 5:586–595, 1992.
- [9] W.K. Hale. Frequency assignment : theory and applications. *Proc. IEEE*, 60:1497–1514, 1980.

- [10] P.K. Jha. Optimal $L(2, 1)$ -labeling of cartesian products of cycles with an application to independent domination. *IEEE Trans. Circuits & Systems I: Fundamental Theory and Appl.*, 47:1531–1534, 2000.
- [11] P.K. Jha, A. Narayanan, P. Sood, K. Sundaram, and V. Sunder. On $L(2, 1)$ -labeling of the cartesian product of a cycle and a path. *Ars Combin.*, 55:81–89, 2000.
- [12] B.H. Metzger. Spectrum management technique. Paper presented at the 38th National ORSA Meeting, Detroit, MI, 1970.
- [13] M. Molloy and M.R. Salavatipour. Frequency channel assignment on planar networks. In *In Proc. 10th Annual European Symposium (ESA 2002), Rome, Italy, September 2002*, volume 2461, pages 736–747. Lecture Notes Computer Science, Springer-Verlag Berlin, 2002.
- [14] D. Sakai. Labeling chordal graphs with a condition at distance two. *SIAM J. Discrete Math.*, 7:133–140, 1994.
- [15] J. van den Heuvel. Radio channel assignment on 2-dimensional lattices. *Annals of Combinatorics*, 6:463–477, 2002.
- [16] J. van den Heuvel, R. A. Leese, and M. A. Shepherd. Graph labelling and radio channel assignment. *Journal of Graph Theory*, 29:263–284, 1998.
- [17] M. Whittlesey, J. Georges, and D.W. Mauro. On the λ -number of Q_n and related graphs. *SIAM J. Discrete Math.*, 8:499–506, 1995.