No-Hole $L(p,0)$ Labelings[†]

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We address here a particular case of the general problem of λ labelings, motivated by frequency assignment for telecommunication networks. In this model, stations lying within a given radius r must use frequencies that differ at least by a value p, while stations that are within a larger radius r' must use frequencies that differ by at least another value q. The aim is to minimize the span of frequencies used in the network. This problem can be modeled by a graph labeling problem, called the $L(p,q)$ labeling, where one wants to label nodes of a graph G modeling the network by integers in the range $[0;M]$, while minimizing the value of M. M is then called the λ number of G, and is denoted by $\lambda_q^p(G)$.
A variant of this problem is when all the possible frequencies (i.e., all the possible values in the span) must be used. In this paper, we focus on this problem. More precisely, the constraints are that: (1) all the frequencies are used and (2) condition (1) being satisfied, the span must be minimum. This problem is usually called the *no-hole* $L(p,q)$ labeling problem. Let $[0;M']$ be this new span (thus, clearly, $M' \geq M$). We then call the ν number of G the value M', and we denote it by $\nu_q^p(G)$. Hence, we have $\nu_q^p(G) \geq \lambda_q^p(G)$ for any p,d and G (a no-hole $L(p,q)$ labeling being an $L(p,q)$ labeling). We study here a special case of no-hole $L(p,q)$ labeling, where $q=0$. We first give some general results, before focusing on some specific topologies, namely: cycles, d-dimensional hypercubes, 2-dimensional and 3-dimensional grids. For each of those topologies, we give bounds on the ν_0^p number for any value of $p \geq 1$, and show optimality in some cases.

Keywords: Frequency assignement problem, $L(p,q)$ labeling, no-hole labeling.

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1 Preliminaries

In this paper, we study the *frequency assignment problem*, that arises in wireless communication systems. We are interested here in minimizing the number of frequencies used in the framework where radio transmitters that are geographically close may interfere if they are assigned close frequencies. This problem has originally been introduced in [Met70] and later developed in [Hal80], where radio transmitters that are geographically close may interfere if they are assigned close frequencies. This problem has been shown to be equivalent to a graph labeling problem, in which the nodes represent the transmitters, and any edge joins two transmitters that are suffi ciently close to potentially interfere. The aim here is to label (i.e. give an integer value, corresponding to the frequency) the nodes of the graph in such a way that:

- any two neighbors (transmitters that are very close) are assigned labels (frequencies) that differ by a parameter at least p ;
- any two nodes at distance 2 (transmitters that are close) are assigned labels (frequencies) that differ by a parameter at least q ;

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- the gap between the smallest and the greatest value for the labels is minimized.

It has been proved that under this model, we could assume the labels to be integers, starting at 0 [GY92]. In that case, the minimum range of frequencies that is necessary to assign to the nodes of a graph G is denoted $\lambda_q^p(G)$, and the problem itself is usually called the $L(p,q)$ labeling problem. This frequency assignment problem has been studied in many different specific topologies [GY92, Sak94, WGM95, BPT00, BKTvL00, CKK+02, MS02, BPT02]. The case $p = 2$ and $q = 1$ is the most widely studied (see for instance [CK96, JNS+00, Jha00, CP01]). Some variants of the model also exist, such as the following generalization where one gives k constraints on the k first distances (any two nodes at distance $1 \leq i \leq k$ in G must be assigned labels differing by at least δ_i). One of the issues also considered is the *no-hole labeling*, where one wants to use *all* the frequencies in the span. More precisely, we want that: (a) all the frequencies are used and (b) condition (a) being satisfied, the span must be minimum. Let $[0;M']$ be the span of frequencies that we obtain. We then call the ν number of G the value M', and we denote it by $\nu_q^p(G)$. Hence, we clearly have $\nu_q^p(G) \geq \lambda_q^p(G)$ for any p, d and G.
In this paper, we study a special case of no-hole $L(p,q)$ labeling, namely where $q = 0$. We also focus on a number of specific topologies: cycles, 2-dimensional and 3-dimensional grids and hypercubes of dimension d, H_d. For each of the mentioned topologies cited above, we give bounds on the ν_0^p number for any value of $p \geq 1$ and $d \geq 1$, and show optimality in some cases.

2 No-Hole $L(p,0)$ Labelings

Proposition 1 (General Graphs) *For any $p \geq 1$ and any connected graph G, if a no-hole labeling of G exists, then:*

- $\nu_0^p(G) \geq 2p-1$ *if G is bipartite ;*
- $\nu_0^p(G) \geq 2p$ *if G is not bipartite.*

Proof: In any graph such that a no-hole $L(p,0)$ labeling exists, there must, by definition, exist at least one node of label 0. Let u be this node. Then all the neighbors of u must be labeled at least p. Since the labeling is no-hole, all the labels in the range $[0;p]$ must be used. This is true, in particular, for label $p-1$. Let v be a node whose label is $p-1$. Then v has all its neighbors labeled at least $2p-1$.
Now, let G be non bipartite, and suppose that $\nu_0^p(G) \leq 2p-1$. G has at least an odd cycle. Consider the nodes on this cycle, and let $i \geq 0$ be the minimum label among them, assigned to node v. If $i \geq p$, then the neighbors of v, not being of minimum label, must be assigned a label at least $2p$, a contradiction. Hence, $i \in [0;p-1]$. In that case, the two neighbors of v on the cycle, say w_1 and w_2, are assigned labels at least $p+i$, that is in the range $[p;2p-1]$. But the neighbors of w_1 and w_2 on the cycle must be assigned labels in the range $[0;p-1]$, etc. If we repeat this argument, we see that, when we will close the cycle, since it is odd, we will end up with a node z whose two neighbors, say x and y, are such that x is assigned a label in the range $[0;p-1]$, while y is assigned a label in the range $[p;2p-1]$. If $\nu_0^p(G) \geq 2p-1$, there is no possibility to label z, a contradiction. □

Observation 1 *For any $p,q \geq 0$ and any graph G of order n that admits a no-hole labeling, $\nu_q^p(G) \leq n-1$.*

Proof: Suppose a no-hole $L(p,q)$ labeling for G exists. In order to be able to assign the nodes of G all the labels in the range $[0;\nu_q^p(G)]$, we must have $n \geq \nu_q^p(G)+1$. □

Proposition 2 (Cycles) *For any $p \geq 1$ and any graph G:*

- $\nu_0^p(C_n) = 2p$ *for any odd $n \geq 2p+1$;*
- $\nu_0^p(C_n) = 2p-1$ *for any even $n \geq 2p+2$.*

Proposition 3 (Hypercubes) *for any $d \geq \frac{p+4}{2}$, $\nu_0^p(H_d) = 2p-1$.*

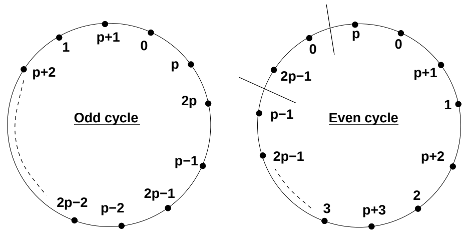

Fig. 1: $L(p,0)$ no-hole labeling in cycles

Sketch of Proof: We first know that $\nu_0^p(H_d) \geq 2p-1$ by Proposition 1. We will first show that $\nu_0^p(H_d) \leq 2p-1$ (thus, proving the equality) for any $d \geq 2p-1$; then, we will show that this result can be extended to any $d \geq \frac{p+4}{2}$. Suppose $d \geq 2p-1$. The fact that $\nu_0^p(H_d) \leq 2p-1$ is proved by homomorphism into the following graph G': (a) the nodes of G' are the integers between 0 and $2p-1$ and (b) there is an edge between u and v in G' iff $|u-v| \geq p$. Clearly, G' represents the constraints on the $L(p,0)$ labeling. We want to achieve an homomorphism $\mathcal{H}$ from H_d to G', that is we want to find a mapping from $V(H_d)$ to $V(G')$, where every node v has an image $h(v)$ such that any edge (u,v) in H_d corresponds to an edge $(h(u),h(v))$ in G'. If we can do this, then we can find a labeling (more precisely, $c(v) = h(v)$ for any node v) that satisfies the $L(p,0)$ constraint. But since we also want this labeling to be no-hole, we also need that every node of G' is image of at least one node of H_d. Here, we take H_d and partition its nodes into $d+1$ sets: for any $0 \leq i \leq d$, the set S_i corresponds to the nodes having exactly i bits equal to 0 in its binary coordinates. By definition of the hypercube, it can be seen that for every $0 \leq i \leq d$, S_i is a stable set. In other words, all the edges appear between different S_is. More precisely, all the edges of H_d appear between an S_i and an S_{i+1}. Now, in the homomorphism we want to achieve, we set that all nodes belonging to the same S_i have the same image by $\mathcal{H}$. Let h_i be the image by $\mathcal{H}$ of all the nodes of S_i. Then, for any $1 \leq i \leq d-1$, h_i must be connected in G'_p to both h_{i+1} and h_{i-1}. Moreover, h_0 must be connected to h_1, and h_{d-1} must be connected to h_d. Hence, this induces a path starting at h_0, and ending at h_d, with edges (h_i,h_{i+1}) for any $0 \leq i \leq d-1$. But we also want this labeling to be no-hole, hence this path must be hamiltonian. In other words, if we are able to find a hamiltonian path in G'_p, then there exists a homomorphism of H_d into G'_p. Clearly, since we have $d+1$ sets S_i, and since each one has a unique image in G'_d, we must have $d+1 \geq 2p$, that is $d \geq 2p-1$.

Now we will show that G'_p contains a hamiltonian path ; it is as follows: $p, 0, p+1, 1 \ldots i, p+i, i+1, p+i+1, \ldots p-2, 2p-2, p-1, 2p-1$ (cf. Figure 2(left)).

Let v_j be any node of set S_j, $0 \leq j \leq d$. If $j = 2i$, we set $h(v_{2i}) = p+i$ for every $0 \leq i \leq p-1$ and if $j = 2i+1$, we set $h(v_{2i+1}) = i$ for every $0 \leq i \leq p-1$. Finally, for any $j \geq 2p$, if j is of the form $2p+2i$, we set $h(v_j) = 2p-1$, and if j is of the form $2p+2i+1$, we set $h(v_j) = 0$.

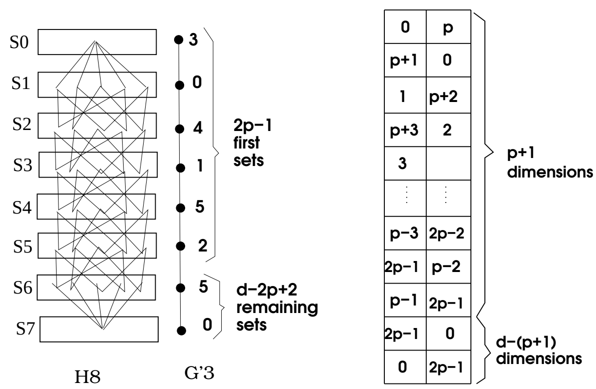

Fig. 2: (left) Homomorphism of H_d into G'_p, with $d = 7$ and $p = 3$; (right) Homomorphism of H_d into $G(2,d)$, where nodes of $G(2,d)$ are represented by squares

We can show that the above result can be extended for any $d \geq p+1$. This is obtained using the same kind of argument (that is, homomorphism into G'_p), but with a better mapping of the nodes (cf. Figure 2(right)). The same goes to extending the result for any $d \geq \frac{p+4}{2}$, hence proving the proposition. Here again, the proof technique is the same as for the two previous cases. Roughly, we now consider H_d as 4 copies of H_{d-2}, connected between them by 2 perfect matchings. □

Proposition 4 *If $\nu_0^p(H_{d_0}) = 2p-1$ for a given dimension d_0, then $\nu_0^p(H_{d'}) = 2p-1$ for any $d' \geq d_0$.*

Sketch of Proof: Consider H_{d_0}, for which we have a no-hole $L(p,0)$ labeling with $\nu_0^p(H_{d_0}) = 2p-1$. We can show a way to obtain a no-hole $L(p,0)$ labeling of H_{d_0+1} from the labeling of H_{d_0}: indeed, H_{d_0+1} is obtained from two copies of H_{d_0} joined by a perfect matching. Thus, we take two copies of an already labeled H_{d_0}, and we fi nd a way to match each node of the fi rst copy to each node of the second copy, such that the labeling remains $L(p,0)$. □

Proposition 5 (2-Dimensional Grids $P_n \times P_m$) *For any $n \geq m \geq 1$ and $1 \leq p \leq \frac{n \cdot m - m}{2}$, $\nu_0^p(P_n \times P_m) = 2p-1$.*

Sketch of Proof: For any $n \geq m$, $P_n \times P_m$ is bipartite, thus by Proposition 1, we have $\nu_0^p(P_n \times P_m) \geq 2p-1$. Moreover, there is a way to put the labels of the range $[0; p-1]$ on one bipartition, and the labels of the range $[p; 2p-1]$ on the other, as shown in Figure 3, such that the labeling is $L(p,0)$. Since m labels (among the $n \cdot m$ vertices of $P_n \times P_m$) are repeated, we need to have $2p \geq n \cdot m - m$ in order for this labeling to be no-hole. Hence, in that case we have $\nu_0^p(P_n \times P_m) \leq 2p-1$ and the equality holds. □

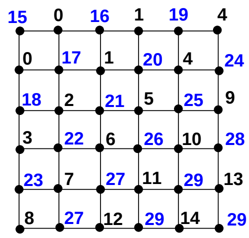

Fig. 3: An optimal no-hole $L(15,0)$ labeling of $P_6 \times P_6$ with $\nu_0^{15} = 29$

We can extend the previous result to 3-dimensional grids (a particular example is given in Figure 4).

Proposition 6 (3-Dimensional Grids $P_n \times P_m \times P_k$) *For any $n \geq m \geq k \geq 1$ and $1 \leq p \leq \frac{n \cdot m \cdot k - m \cdot k}{2}$, $\nu_0^p(P_n \times P_m \times P_k) = 2p-1$.*

24	0	25	1
0	26	1	31
27	2	32	7
3	33	8	43

0	28	1	34
29	2	35	7
3	36	8	44
37	9	45	17

30	4	38	10
5	39	11	46
40	12	47	18
13	47	19	47

6	41	14	47
42	15	47	20
16	47	21	47
47	22	47	23

Fig. 4: An optimal $L(24,0)$ no-hole labeling in $P_4 \times P_4 \times P_4$ with $\nu_0^{24} = 47$. Each of the 4 blocks represents a 2-Dimensional subgrid $P_4 \times P_4$, in which each square represents a node.

3 $L(p,q)$ Labelings

We conclude this paper by giving new results concerning the (general, i.e. not necessarily no-hole) $L(p,q)$ labeling of H_d. Some results have been given in [FR03]. However, it is possible to improve them.

Proposition 7 *For any $p, q \geq 0$ and $d \geq 1$:*

(1) $\lambda_q^0(H_d) \leq (2d-3)q$ *;*

(2) $\lambda_q^p(H_d) \leq 2p+(2d-2)q-1$.

Sketch of Proof: In both cases, we give a coloring that satisfies the rules and do not exceed the given bound. More precisely, for any vertex $v=(x_1,x_2\ldots x_d)$ of H_d, we define : in case (1), $c(v) = (\sum_{i=1}^{d-1} iqx_i)\ mod\ (2d-2)q$ and in case (2), $c(v) = \sum_{i=1}^{d}(p+(i-1)q)x_i\ mod\ (2p+(2d-2)q)$ □

We note that item (2) of Proposition 7, when applied to the case $p=2$ and $q=1$, gives $\lambda_q^p(H_d) \leq 2d+1$, which is the lower bound proved in [GY92].

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