New Bounds for the $L(h,k)$ Number of Regular Grids

Tiziana Calamoneri
Dipartimento di Informatica Università degli Studi di Roma “La Sapienza”
Via Salaria 113, 00198 Roma, Italy
E-mail: calamo@di.uniroma1.it

Saverio Caminiti
Dipartimento di Informatica, Università degli Studi di Roma “La Sapienza”
Via Salaria 113, 00198 Roma, Italy
E-mail: caminiti@di.uniroma1.it

Guillaume Fertin*
Laboratoire d’Informatique de Nantes-Atlantique
FRE CNRS 2729, Université de Nantes
2 rue de la Houssinière, BP 92208 - 44322 Nantes Cedex 3, France
E-mail: fertin@lina.univ-nantes.fr
*Corresponding author

Abstract: For any non negative real values h and k, an $L(h,k)$-*labeling* of a graph $G=(V,E)$ is a function $L : V \rightarrow \mathbb{R}$ such that $|L(u)-L(v)| \geq h$ if $(u,v) \in E$ and $|L(u)-L(v)| \geq k$ if there exists $w \in V$ such that $(u,w) \in E$ and $(w,v) \in E$. The *span* of an $L(h,k)$-labeling is the difference between the largest and the smallest value of L. We denote by $\lambda_{h,k}(G)$ the smallest real λ such that graph G has an $L(h,k)$-labeling of span λ. The aim of the $L(h,k)$-labeling problem is to satisfy the distance constraints using the minimum span.
In this paper, we study the $L(h,k)$-labeling problem on regular grids of degree 3, 4, and 6 for those values of h and k whose $\lambda_{h,k}$ is either not known or not tight. We also initiate the study of the problem for grids of degree 8. For all considered grids, in some cases we provide exact results, while in the other ones we give very close upper and lower bounds.

Keywords: $L(h,k)$-labeling, triangular grids, hexagonal grids, squared grids, octagonal grids

Reference to this paper should be made as follows: Calamoneri, T., Caminiti, S. and Fertin, G. (2006) ‘New Bounds for the L(h,k) Number of Regular Grids’, International Journal of Mobile Network Design and Innovation , Vol. x, No. y, pp.a–b.

Biographical notes: Tiziana Calamoneri is assistant professor at the Department of Computer Science, University of Rome ‘’La Sapienza”. She got her Ph.D. in Computer Science 1997, at University of Rome “La Sapienza,” Italy. Her research interests include parallel and sequential graph algorithms, channel assignment in wireless networks, two- and three-dimensional graph drawing, layout of interconnection networks topologies and optimal routing schemes.

Saverio Caminiti is a PhD student at the Department of Computer Science, University of Rome ”La Sapienza”, Italy. He got his Master Degree in Computer Science in 2003, at University of Rome ”La Sapienza”. His research interests include compact representation of data structures both in sequential and parallel settings, graph algorithms, and graph coloring.

Guillaume Fertin is a Professor at LINA (Laboratoire d’Informatique de Nantes-Atlantique), University of Nantes, France. He completed his PhD degree from University of Bordeaux, France, in 1999. Since 1996, he has been researching in optimization, discrete mathematics and algorithmics, with two main applications fields: interconnection networks and bioinformatics.

1 INTRODUCTION

In this paper, we are interested in the *frequency assignment problem*, that arises in wireless communication systems. More precisely, we focus here on minimizing the number of frequencies used in the framework where radio transmitters that are geographically close may interfere if they are assigned close frequencies. This problem has originally been introduced in (17) and was later developed in (12). It is equivalent to a graph labeling problem, in which the nodes represent the transmitters, and any edge joins two transmitters that are sufficiently close to potentially interfere. The aim here is to label the nodes of the graph in such a way that:

- any two neighbors (transmitters that are very close) are assigned labels (frequencies) that differ by a parameter at least h ;
- any two nodes at distance 2 (transmitters that are close) are assigned labels (frequencies) that differ by a parameter at least k ;
- the gap between the smallest and the greatest value for the labels is minimized.

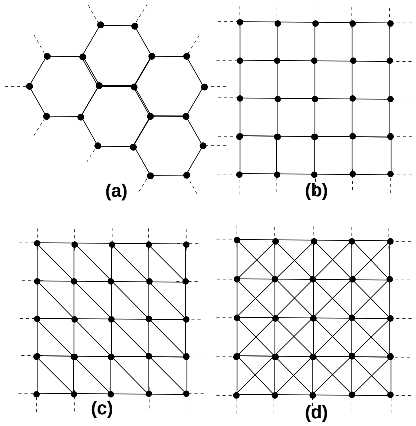
This problem is usually referred to as the $L(h,k)$-labeling problem. More formally, for any non negative real values h and k, an *$L(h,k)$-labeling* of a graph $G = (V,E)$ is a function $L : V \to \mathbb{R}$ such that $|L(u) - L(v)| \geq h$ if $(u,v) \in E$ and $|L(u) - L(v)| \geq k$ if there exists $w \in V$ such that $(u,w) \in E$ and $(w,v) \in E$. The *span* of an $L(h,k)$-labeling is the difference between the largest and the smallest value of L. Hence, it is not restrictive to assume 0 as the smallest value of L, something which will be assumed throughout this paper. We denote by $\lambda_{h,k}(G)$ the smallest real λ such that graph G has an $L(h,k)$-labeling of span λ ; we call *$L(h,k)$ number of* G this value. The aim of the $L(h,k)$-labeling problem is to satisfy the distance constraints using the minimum span.

Since its definition (11) as a specialization of the frequency assignment problem in wireless networks (12; 17), the $L(h,k)$-labeling problem has been intensively studied. Note that the $L(h,k)$-labeling problem is a generalization of some standard graph colorings, such as the usual (or proper) coloring when $h = 1$ and $k = 0$, or the 2-distance coloring (equivalent to the proper coloring of the square of the graph) when $h = k = 1$. We also note that the case $h = 2$ and $k = 1$ (or, more generally $h = 2k$), called radio-coloring or λ-coloring, is the most widely studied (see for instance (7; 9; 13; 14)).

The decision version of the $L(h,k)$-labeling problem is NP-complete even for small values of h and k (2). This motivates seeking optimal solutions on particular classes of graphs (see for instance (3; 4; 8; 11; 18; 19; 20; 15) and (6) for a complete survey). Concerning the more specific grid topologies, a large number of papers has been published on the subject. For instance, Makansi (16) provided an optimal $L(0,1)$-labeling for squared grids, that is regular grids of degree 4 (see Figure (b)). Battiti, Bertossi and Bonuccelli (1) found an optimal $L(1,1)$-labeling for hexagonal, squared and triangular grids (that is, respectively, regular grids of degree 3, 4 and 6, see Figures (a), (b) and (c)). The $L(2,1)$-labeling problem of regular grids of degree Δ, denoted G_Δ, has been studied independently by different authors (3; 7) proving that $\lambda_{2,1}(G_\Delta) = \Delta + 2$ by means of optimal coloring algorithms. More recently, Fertin and Raspaud (10) determined several bounds on $\lambda_{h,k}$ for d-dimensional squared grids.

In (5) some values of $\lambda_{h,k}$ for regular grids of degree 3, 4, and 6 are exactly computed, while in some intervals different upper and lower bounds are given ; moreover, the case $h < k$ is not considered at all. Our goal in this paper is to improve some of those bounds, as well as to consider the case $h < k$. Moreover, we extend this study to a new class of graphs, namely grids of degree 8. Grids of degree 8 can be defined as the strong product of two infinite paths (15) (see also Figure for a graphical representation of the four types of grids we study in this paper). Grids of degree 8 can also be seen as a natural extension of grids of degree 6, who themselves are an extension of grids of degree 4 (see Figures (a), (b) and (c)).

Figure 1 *Grids studied in this paper: (a) G_3, (b) G_4, (c) G_6 and (d) G_8*



Before going further, we observe that when $h < k$ (a case that we will consider in this paper), there are actually two ways to define the $L(h,k)$-labeling problem:

- The first one is the *distance-based* model, which asks that two *neighbors* in the graph differ by at least h, while two nodes *at distance 2* differ by at least k. This means that when two nodes are at the same time connected by a 1-path and a 2-path (hence when there is a cycle of length 3 in the graph), we consider the distance to be 1, and thus impose only the condition on h.
- The second one is the *max-based* model, which asks that two nodes connected at the same time by a 1-path and a 2-path differ by at least $\max\{h,k\}$; in that

sense, this model is more restrictive than the *distance-based* model. In particular, this model imposes that any cycle of length 3 to be always labeled with three labels at least $\max\{h,k\}$ apart from each other.

Note that when $h \geq k$, the two definitions coincide, since $\max\{h,k\} = h$. The same occurs when the considered graph has no triangles, which is the case for G_3 and G_4. In this paper, in the study of G_6 and G_8, when $h < k$, we chose to consider the *max-based* problem.
As mentioned above, we study in this paper the $L(h,k)$-labeling problem on regular grids of degree 3, 4, and 6 for those values of h and k whose $\lambda_{h,k}$ is either not known or not tight, and we also study the $L(h,k)$ labeling problem in a new class of graphs, namely grids of degree 8. For all considered grids, in some cases we provide exact results, or we give close upper and lower bounds (see Figure 6.2 at the end of the paper for a summary of results).
The paper is organized as follows: in Section , we give a few technical lemmas that will help to obtain general lower and upper bounds for the considered types of graphs, while in Sections , 3.2, 4.2 and 4.2, we improve bounds on the $L(h,k)$ number of grids for degree 3, 4, 6 and 8, respectively.
Note finally that if no confusion arises, we will speak interchangeably, in the rest of this paper, of a node and its label.

2 PRELIMINARIES

In this section, we show four different lemmas, which will prove to be useful in the rest of the paper. Lemmas 1 and 1 are concerned with lower bounds for the $L(h,k)$ number, while Lemmas 2 and 3 deal with upper bounds.

Theorem 1. *$\lambda_{h,k}(G_\Delta) \geq h + (\Delta - 1)k$ when $h \leq k$, for $\Delta = 3, 4$.*

Proof. Consider an optimal $L(h,k)$-labeling of G_Δ, $h \leq k$, $\Delta = 3, 4$, and let x be a node labeled 0. The smallest label among those of its neighbors must be at least h. Furthermore, the Δ neighbors of x are all connected by a 2-length path and hence their labels must differ by at least k from each other. It follows that the greatest label must be at least $h + (\Delta - 1)k$. ☐

Lemma 1. *$\lambda_{h,k}(G_\Delta) \geq \Delta k$ when $h \leq k$, for $\Delta = 6, 8$.*

Proof. Observe that G_6 and G_8 are characterized by the property that each pair of adjacent nodes is also connected by a 2-length path. This implies that, given an optimal $L(h,k)$-labeling of G_Δ, $h \leq k$, $\Delta = 6, 8$, starting from a node x labeled 0, the smallest label, among those of their neighbors must be at least k. With reasonings analogous to those of the previous proof, the claim follows. ☐

Lemma 2. *For any graph G and any $h \leq k$, $\lambda_{h,k}(G) \leq k \cdot \lambda_{1,1}(G)$.*

Proof. Consider an optimal $L(1,1)$-labeling, say $\mathcal{L}$, of G. Consider the labeling $\mathcal{L}'$ obtained from $\mathcal{L}$ by substituting every label i with label ik ($i = 0, 1, \ldots, \lambda_{1,1}(G)$). We claim that $\mathcal{L}'$ is an $L(h,k)$-labeling of G with span $k \cdot \lambda_{1,1}(G)$, provided $h \leq k$. Indeed, any two neighbors, which differ by at least 1 in $\mathcal{L}$, differ by at least $k \geq h$ in $\mathcal{L}'$; moreover, any two nodes connected by a 2-length path, which differ by at least 1 in $\mathcal{L}$ differ by at least k in $\mathcal{L}'$. ☐

Lemma 3. *For any graph G and any $h \geq \frac{k}{2}$, $\lambda_{h,k}(G) \leq h \cdot \lambda_{1,2}(G)$.*

Proof. Analogously to the proof of Lemma 2, consider an $L(1,2)$ labeling, say $\mathcal{L}$, of G. Consider the labeling $\mathcal{L}'$ obtained from $\mathcal{L}$ by substituting every label i with label ih ($i = 0, 1, \ldots, \lambda_{1,2}(G)$). Since $h \geq \frac{k}{2}$, $\mathcal{L}'$ is an $L(h,k)$-labeling of G with span $h \cdot \lambda_{1,2}(G)$. Indeed, any two neighbors, which differ by at least 1 in $\mathcal{L}$, differ by at least h in $\mathcal{L}'$; moreover, any two nodes connected by a 2-length path, which differ by at least 2 in $\mathcal{L}$ differ by at least $2h \geq k$ in $\mathcal{L}'$. ☐

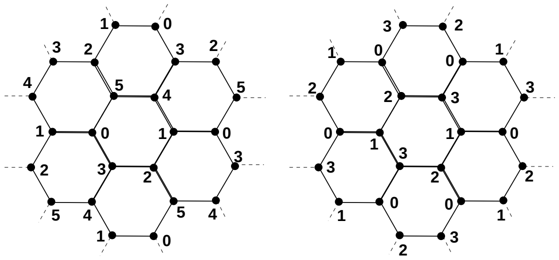
3 REGULAR GRIDS OF DEGREE 3

3.1 Upper Bounds for G_3

Proposition 1. *$\lambda_{h,k}(G_3) \leq h + 2k$ when $h \leq \frac{k}{2}$.*

Proof. Consider an optimal $L(1,2)$-labeling of G_3 over the set of labels $\{0, 1, \ldots, 5\}$, whose general pattern is depicted in Figure 3.1(a). The idea is to substitute h to 1, k to 2, $h+k$ to 3, $2k$ to 4, and $h+2k$ to 5. In that case, the labeling that is produced is a feasible $L(h,k)$-labeling. Indeed, each pair of consecutive labels differs by either h or $k-h$, but since we supposed $h \leq \frac{k}{2}$, we have $k - h \geq h$ and thus any two consecutive labels differ by at least h. Similarly, any other pair of distinct labels differs by at least k. Moreover, the largest label used is $h + 2k$, hence the result. ☐

Figure 2 *General patterns for $L(h,k)$-labelings of G_3: (a) $L(1,2)$-labeling ; (b) $L(1,1)$-labeling*



(a) (b)

Proposition 2. *$\lambda_{h,k}(G_3) \leq min\ \{5h, 3k\}$ when $\frac{k}{2} \leq h \leq k$.*

Proof. By Lemma 3, since $\frac{k}{2} \leq h$ and since there exists an $L(1,2)$-labeling of G_3 that is of span 5 (see for instance the general pattern shown in Figure 3.1(a)), we know there exists an $L(h,k)$-labeling of G_3 of span $5h$.

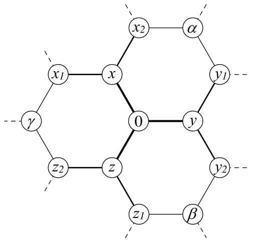
Analogously, since $h \leq k$, we obtain an $L(h,k)$-labeling of span $3k$ by Lemma 2 ; indeed, there exists an $L(1,1)$-labeling of G_3 that is of span 3 (whose general pattern is shown in Figure 3.1(b), see also (1)). ☐

3.2 Lower Bounds for G_3

Proposition 3. $\lambda_{h,k}(G_3) \geq h + 2k$ *when* $h \leq k$.

Proof. This bound directly comes from Lemma 1. ☐

Figure 3 *Neighborhood of a node labeled 0 in G_3*



Proposition 4. $\lambda_{h,k}(G_3) \geq 3k$ *when* $\frac{2k}{3} \leq h \leq k$.

Proof. Consider an optimal $L(h,k)$-labeling of G_3. Suppose, by contradiction, that $\lambda_{h,k}(G_3) < 3k$. Let us consider a node labeled 0, and let x, y, and z be its 3 neighbors. Without loss of generality, suppose $x < y < z$. In view of the $L(h,k)$-constraints, we must have $x \geq h$, $y \geq x + k \geq h + k$, and $z \geq y + k \geq h + 2k$. Furthermore, from the hypothesis $\lambda_{h,k}(G_3) < 3k$, we have that $z < 3k$, hence $y \leq z - k < 2k$, and $x \leq y - k < k$. Let x_1 and x_2, y_1 and y_2, z_1 and z_2 be the not 0 neighbors of x, y, and z, respectively (see Figure 3.2).

Let us first prove that if $y_m = \min\{y_1, y_2\}$ and $y_M = \max\{y_1, y_2\}$, then $y_m < y < y_M$. Indeed, if $y < y_m$, then $y_m \geq y + h \geq 2h + k$, and consequently $y_M \geq 2h + 2k$. However, $2h + 2k \geq 3k$ (because we supposed $h \geq \frac{2k}{3} \geq \frac{k}{2}$), a contradiction to the fact that $\lambda < 3k$. On the other hand, if $y_M < y$, then $y \geq y_M + h$. And since $y_M \geq y_m + k \geq 2k$, we end up with $y \geq h + 2k$. However, by hypothesis we know that $y < 2k$, a contradiction since $h \geq 0$. Thus we conclude that in all the cases, we have $y_m < y < y_M$.

Now, in order to prove the statement, we will show that under the hypothesis $\lambda_{h,k}(G_3) < 3k$, both cases $x_1 < x_2$ and $x_1 > x_2$ lead to a contradiction.

Case 1: $x_1 < x_2$. In this case $x_1 \geq k$, as x_1 is connected by a 2-length path to node 0 (via x) and $x_2 \geq x_1 + k \geq 2k$. If $x_1 < x$, then $x \geq x_1 + h \geq k + h$, a contradiction since $x < k$. Hence, $x < x_1 < x_2$. It follows that $x_1 \geq x + h \geq 2h$ and $x_2 \geq x_1 + k \geq 2h + k$. Let us now consider y_1 and y_2.

Case 1.1: $y_1 < y_2$. Hence we know that $y_1 < y < y_2$. In such a case $y_1 \geq k$ and $y_1 \leq y - h < 2k - h$. Note that $y_1 < x_2$ as $y_1 < 2k - h$ and $x_2 \geq 2k$. Let us consider the common neighbor of x_2 and y_1, α, and let us study the relative position of its label with respect to x_2 and y_1.

- $\alpha < y_1 < x_2$. Then $\alpha \leq y - k < k$: if $x < \alpha$ we have $\alpha \geq x + k \geq h + k$, a contradiction ; on the other hand, if $\alpha < x$ then $\alpha \leq x - k < 0$, a contradiction too.
- $y_1 < x_2 < \alpha$. Then $x_2 \leq \alpha - h < 3k - h$; from previous hypotheses we also have $x_2 \geq 2h + k$, and this leads to a contradiction as $3k - h \leq 2h + k$ when $h \geq \frac{2k}{3}$.
- $y_1 < \alpha < x_2$. We have again two cases. If $y_1 < \alpha < y$ then $\alpha \leq y - k < k$ and $y_1 \leq \alpha - h < k - h$ that is a contradiction as $y_1 \geq k$. If $y_1 < y < \alpha$ then $\alpha \leq x_2 - h < 3k - h$, $y \leq \alpha - k < 2k - h$, and $y_1 \leq y - h < 2k - 2h$ that is a contradiction as $y_1 \geq k$ and $k \geq 2k - 2h$ when $\frac{2k}{3} \leq h \leq k$.

Case 1.2: $y_1 > y_2$. Thus we have $y_1 > y > y_2$. This implies that $y_1 \geq y + h \geq 2h + k$. Hence, y_1 lies in the interval $[2h + k; 3k[$. However, we also know that x_2 lies in the interval $[2h + k; 3k[$. Since this interval is of width $w < 2k - 2h$, we conclude that $w < k$ (because we supposed $h \geq \frac{2k}{3}$ and hence $h \geq \frac{k}{2}$). This leads to a contradiction because y_1 and x_2 must be at least k away from each other.

Case 2: $x_1 > x_2$. With considerations analogous to those done for case $x_1 < x_2$, we can derive $x < x_2 < x_1$ and $2h + k \leq x_1 < 3k$ and $2h \leq x_2 < 2k$. Now, let us look at y_1 and y_2.

Case 2.1: $y_1 < y_2$. We thus have $y_1 < y < y_2$. However, this leads to a contradiction. Indeed, $y_1 > k$ as it is connected by a 2-length path to node 0, then $x_2 \geq y_1 + k > 2k$.

Case 2.2: $y_1 > y_2$. We then have $y_2 < y < y_1$. This implies that $y_1 \geq y + h \geq 2h + k$ and hence $y_1 > x_2$ as $x_2 < 2k$. Now consider α, the common neighbor of x_2 and y_1.

- $x_2 < y_1 < \alpha$. Then $\alpha \geq y_1 + h \geq 3h + k \geq 3k$, a contradiction since we supposed $\lambda < 3k$.
- $\alpha < x_2 < y_1$. Then $\alpha \leq x_2 - h < 2k - h$. If $\alpha > y$ then $\alpha \geq y + k \geq h + 2k$, a contradiction ; if $\alpha < y$ then $\alpha \leq y - k \leq k$. However, we know that $x < k$; moreover, because $\alpha < k$ and α must lie at least k away from x, this leads to a contradiction.
- $x_2 < \alpha < y_1$. Then $\alpha \leq y_1 - h < 3k - h$. If $\alpha > y$ then $\alpha \geq y + k \geq h + 2k$ that is greater than $3k - h$ under the hypothesis $h \geq \frac{2k}{3}$, a contradiction ; if $\alpha < y$ then $\alpha \leq y - k \leq k$ that again contradicts the fact that α must lie at least k away from x.

Altogether, we see that every possible case leads to a contradiction. This proves that the initial assumption, $\lambda < 3k$, is false, and consequently the proposition is proved. ☐

Proposition 5. *$\lambda_{h,k}(G_3) \geq 3h$ when $k \leq h \leq \frac{3k}{2}$.*

Proof. The proof is analogous to the previous one, i.e., by contradiction we assume that there exists a $L(h,k)$-labeling with span $\lambda < 3h$, we start from node labeled 0, we look at its neighbors and prove that neither $x_1 < x_2$ nor $x_1 > x_2$ can occur. Wlog, let us assume $x < y < z$. Hence, $x \geq h$, $y \geq h+k$ and $z \geq h+2k$. On the other hand, $z < 3h$, $y < 3h-k$ and $x < 3h-2k$. Let x_1 and x_2, y_1 and y_2, z_1 and z_2 be the not 0 neighbors of x, y, and z, respectively (see Figure 3.2).

We first prove that if $y_m = \min\{y_1, y_2\}$ and $y_M = \max\{y_1, y_2\}$, then $y_m < y < y_M$. Indeed, if $y < y_m$, then $y_m \geq y+h \geq 2h+k$, and consequently $y_M \geq 2h+2k$. However, $2h+2k \geq 3h$ (because we supposed $h \leq \frac{3k}{2}$), a contradiction to the fact that $\lambda < 3h$. On the other hand, if $y_M < y$, then $y \geq y_M + h$. And since $y_M \geq y_m + k \geq 2k$, we end up with $y \geq h+2k$. However, by hypothesis we know that $y < 3h-k$, a contradiction since $3h-k \leq h+2k$, because we supposed $h \leq \frac{3k}{2}$. Thus we conclude that in all the cases, we have $y_m < y < y_M$. Now, as in the previous proof, let us consider x_1 and x_2 (see Figure 3.2), and show that, under the hypothesis $\lambda < 3h$, none of the cases $x_1 < x_2$ and $x_1 > x_2$ can occur.

Case 1: $x_1 < x_2$. This implies $x_1 \geq k$, as x_1 is connected by a 2-length path to node 0 (via x). If $x_1 < x$, then $x \geq x_1 + h \geq h+k$, that is a contradiction as $x < 3h-2k \leq h+k$ under the hypothesis $h \leq \frac{3k}{2}$. Hence, $x < x_1 < x_2$. It follows that $x_1 \geq x+h \geq 2h$ and $x_2 \geq x_1 + k \geq 2h+k$. Let us consider now y_1 and y_2.

Case 1.1: $y_1 < y_2$. Then we know that $y_1 < y < y_2$. Note that $y_1 < x_2$ as $x_2 \geq 2h+k$ and $y_1 \leq y-h \leq y_2-2h < 3h-2h = h$. Now, let us consider α, the common neighbor of y_1 and x_2.

- $y_1 < x_2 < \alpha$. The contradiction comes from the inequality $\alpha \geq x_2 + h \geq 3h+k$.
- $\alpha < y_1 < x_2$. Then $y_1 \geq \alpha + h \geq h$, $y \geq y_1 + h \geq 2h$ and $y_2 \geq y+h \geq 3h$, a contradiction.
- $y_1 < \alpha < x_2$. Since we have $y_1 \geq k$, this implies $\alpha \geq y_1 + h \geq h+k$ and $\alpha \leq x_2 - h < 2h$. It is easy to see that the same bounds hold also for y. Hence y and α both lie in the interval $[h+k; 2h[$, of width $w < h-k$, that is $w \leq k$. The contradiction comes from the fact that α and y being connected by a 2-length path, they must lie at least k away from each other.

Case 1.2: $y_1 > y_2$. Thus, we know that $y_1 > y > y_2$. We know that x_2 and y_1 must be at least k away from each other. Moreover, $2h+k \leq x_2 < 3h$ and $2h+k \leq y_1 < 3h$. Hence, both x_2 and y_1 lie in an interval of width $w < h-k$. Since we supposed $h \leq \frac{3k}{2}$, we conclude $w < k$, a contradiction.

Case 2: $x_1 > x_2$. We can easily see that in that case we must have $x_1 > x_2 > x$. Indeed, $x_2 \geq k$, since it is connected by a 2-length path to node 0. Hence, if $x > x_2$, then $x \geq h+k$. However, we know that $x < 3h-2k$, a contradiction since $h \leq \frac{3k}{2}$. Hence we conclude that $x_1 > x_2 > x$, which implies $x_2 \geq x+h \geq 2h$ and $x_1 \geq x_2 + k \geq 2h+k$. Now let us consider y_1 and y_2.

Case 2.1: $y_1 < y_2$. Let us then consider α, the common neighbor of y_1 and x_2, and let us look at its relative position compared to x and y. There are three possible cases.

- $\alpha > y > x$. We recall that we are in the case $x_1 > x_2 > x$, that is $x_2 \geq x+h \geq 2h$. If $\alpha > x_2$ then $\alpha \geq x_2 + h \geq 3h$, a contradiction to the hypothesis $\lambda < 3h$. Now, if $\alpha < x_2$, $\alpha \leq x_2 - h$. Since $x_2 \leq x_1 - k < 3h-k$, we conclude $\alpha \leq 2h-k$. But $y \geq h+k$ and $\alpha \geq y+k$, that is $\alpha \geq h+2k$. This is a contradiction since $2h-k \leq h+2k$, by the hypothesis that $h \leq \frac{3k}{2}$.
- $y > \alpha > x$. We then conclude that $\alpha \leq y-k < 3h-2k$. On the other hand, we have $\alpha \geq x+k \geq h+k$. This is a contradiction since $h+k \geq 3h-2k$ due to the fact that we supposed $h \leq \frac{3k}{2}$.
- $y > x > \alpha$. In that case, if $\alpha < y_1$, then $y_1 \geq \alpha + h \geq h$, which implies $y \geq 2h$ and $y_2 \geq 3h$, a contradiction to the hypothesis $\lambda < 3h$. Now, if $\alpha > y_1$, then $\alpha \geq h$, which in turns means that $x \geq h+k$ and $y \geq h+2k$. However, we know that $y < 3h-k$, a contradiction since $3h-k \leq h+2k$ due to the fact that we supposed $h \leq \frac{3k}{2}$.

Case 2.2: $y_1 > y_2$. Here, we consider the three nodes z, z_1 and z_2. We first show that if $z_m = \min\{z_1, z_2\}$ and $z_M = \max\{z_1, z_2\}$, then $z_m < z_M < z$. Indeed, if $z_M > z$ then $z_M \geq z+h$, and since we know $z \geq h+2k$, we conclude $z_M \geq 2h+2k$, a contradiction to the fact that $\lambda < 3h$ since $2h+2k \geq 3h$. Now let us look at the relative positions of z_1 and z_2. There are two cases to consider.

- $z_1 > z_2$. In that case, we have $z > z_1 > z_2$. Now let us look at β, common neighbor of z_1 and y_2, and let us consider the relative positions of β and y.
  - $\beta < y$. First, we note that $\beta < z_1$. Indeed, $z_2 \geq k$ (it is connected by a 2-length path to node 0), thus $z_1 \geq 2k$. However, $\beta < y$ by hypothesis, hence $\beta \leq y-k$, that is $\beta < 2h-k$. Moreover, $2h-k \leq 2k$ since we are in the case $h \leq \frac{3k}{2}$, and thus we conclude that $\beta < z_1$. This implies $\beta \leq z_1 - h$, that is $\beta \leq z-2h$; and since $z \leq \lambda < 3h$, we get $\beta < h$. On the other hand, $y_2 < y$, thus $y_2 \leq y-h$. But since $y < 2h$, we then have $y_2 < h$. Hence, both β and y_2 lie in the interval $[0; h[$. However, they are neighbors and thus should have labels that are at least h away, a contradiction.
  - $\beta > y$. Then we have $\beta \geq y+k$, that is $\beta \geq h+2k$. However, we know that $z \geq h+2k$ as

well. Thus, β and z lie in the interval $[h+2k;\lambda[$, where $\lambda < 3h$ by hypothesis. Thus the width of this interval w satisfies $w < 2h-2k$, and thus $w < k$ because we supposed $h \leq \frac{3k}{2}$. However, β and z are neighbors, and thus should have labels at least differing by h, a contradiction with the fact that $w < h$.

- $z_2 > z_1$. In that case, we know that $z > z_2 > z_1$. In particular, this means that $z_2 < 2h$, and $z_1 < 2h-k$. However, $z_1 \geq k$ since it is connected by a 2-length path to node 0. We also have $y \leq z-h < 2h$, and thus $y_2 \leq y-h < h$; and since $h \geq k$, we conclude that $y_2 \leq 2h-k$. Moreover, $y_2 \geq k$ since it is connected by a 2-length path to node 0. Hence, both z_1 and y_2 lie in the interval $[0;2h-k[$, of width $w < 2h-2k$, that is $w < k$ since we supposed $h \leq \frac{3k}{2}$. However, z_1 and y_2 are connected by a 2-length path, and thus should have labels at least differing from k, a contradiction.

Altogether, we see that every possible case leads to a contradiction. This proves that the initial assumption, $\lambda < 3h$, is false, and consequently the proposition is proved. □

Proposition 6. *$\lambda_{h,k}(G_3) \geq h+3k$ when $\frac{3k}{2} \leq h \leq 2k$.*

Proof. Consider an optimal $L(h,k)$-labeling of G_3 with span λ. By contradiction, suppose $\lambda < h+3k$. Let us consider a node labeled 0, and let x, y, and z be its 3 neighbors. Without loss of generality, suppose $x < y < z$. In view of the $L(h,k)$-constraints, we must have $x \geq h$, $y \geq x+k \geq h+k$, and $z \geq y+k \geq h+2k$. Furthermore, for the hypothesis $\lambda < h+3k$, $z < h+3k$, hence $y \leq z-k < h+2k$, and $x \leq y-k < h+k$. Let x_1 and x_2, y_1 and y_2, z_1 and z_2 be the not 0 neighbors of x, y, and z, respectively (see Figure 3.2).
Let us first prove the following, which will be useful in the rest of the proof: if $y_m = \min\{y_1, y_2\}$ and $y_M = \max\{y_1, y_2\}$, then $y_m < y < y_M$. Indeed, if $y < y_m < y_M$, we have $y_m \geq y+h \geq 2h+k$, and $y_M \geq y_m+k \geq 2h+2k$. However, this contradicts the fact that $\lambda < h+3k$, because $2h+2k \geq h+3k$ (since we supposed $h \geq \frac{3k}{2}$). Now suppose $y_m < y_M < y$. Then $y_m \geq k$, because it is connected by a 2-length path to node 0. Thus $y_M \geq y_m+k \geq 2k$, and $y \geq y_M+h \geq h+2k$, which contradicts the fact that $y < h+2k$. Altogether, we conclude that the only possible case is $y_m < y < y_M$ **(1)**.
In the following we show that, under the hypothesis $\lambda < h+3k$, both cases $x_1 < x_2$ and $x_1 > x_2$ lead to a contradiction, which will prove the statement.

Case 1: $x_1 < x_2$. This implies $x_1 \geq k$, as x_1 is connected by a 2-length path to node 0 (via x) and $x_2 \geq x_1+k \geq 2k$. If $x_1 < x$, then $x \geq x_1+h \geq k+h$, that is a contradiction as $x < h+k$. Hence, we have $x < x_1 < x_2$. It follows that $x_1 \geq x+h \geq 2h$ and $x_2 \geq x_1+k \geq 2h+k$. Moreover, $x_1 \leq x_2-k < h+2k$ and $x \leq x_1-h < 2k$. Let us now consider y_1 and y_2.

Case 1.1: $y_1 < y_2$. By **(1)** above, we have $y_1 < y < y_2$. Let us now consider α (common neighbor of y_1 and x_2), and let us study its relative position compared to x and y (we recall that $x < y$ by hypothesis).

- $\alpha > y > x$. Hence we have $\alpha \geq y+k \geq h+2k$. But $x_2 \geq 2h+k \geq h+2k$ as well. Hence, both α and x_2 lie in the interval $[h+2k;h+3k[$, of width $w < k \leq h$. However, x_2 and α are neighbors, thus they must be at least h away, a contradiction.

- $y > \alpha > x$. In that case, $\alpha \leq y-k < 2k$. But we also have $\alpha \geq x+k \geq h+k$, a contradiction.

- $y > x > \alpha$. Since $x < 2k$, we conclude that $\alpha \leq x-k < k$. However, we know $y_1 \geq k$ (because it is connected by a 2-length path to node 0). Thus $\alpha < y_1$, hence $y_1 \geq \alpha+h \geq h$. But we know $y_1 < y < y_2$, thus $y_1 \leq y-h$, and $y \leq y_2-h < 3k$, thus $y_1 < 3k-h$. But we cannot have $y_1 \geq h$ and $y_1 < 3k-h$, since $h \geq \frac{3k}{2}$.

Case 1.2: $y_2 < y_1$. By **(1)** above, we have $y_2 < y < y_1$. Hence $y_1 \geq y+h \geq 2h+k$. We also know that $x_2 \geq 2h+k$, since $x < x_1 < x_2$. Thus y_1 and x_2 share the same interval $[2h+k;h+3k[$, of width $w < 2k-h \leq k$. But y_1 and x_2 are connected by a 2-length path, and thus must be at least k away, which is impossible.

Hence, at this point we conclude that necessarily $x_1 > x_2$. Thus let us consider this case.

Case 2: $x_2 < x_1$. In that case, it is easily seen that actually $x_1 > x_2 > x$, since $x > x_2$ would imply $x \geq x_2+h$; and since $x_2 \geq k$ (it is connected by a 2-length path to node 0), we would have $x \geq h+k$, a contradiction to the fact that $x < h+k$. Now let us look again at the relative positions of y_1 and y_2.

Case 2.1: $y_1 < y_2$. By **(1)** above, we have $y_1 < y < y_2$. This implies that $y \leq y_2-h < 3k$. And since we know by hypothesis that $x < y$, we conclude that $x \leq y-k < 2k$.

- $\alpha > y > x$. Then $\alpha \geq y+k \geq h+2k$. However, we know $x_2 < x_1$, that is $x_2 \leq x_1-k < h+2k$, hence we conclude $\alpha > x_2$. Thus $\alpha \geq x_2+h$, and since $x_2 > x$ we have $x_2 \geq x+h \geq 2h$, we conclude $\alpha \geq 3h$, a contradiction to the fact that $\lambda < h+3k$, since we supposed $h \geq \frac{3k}{2}$.

- $y > \alpha > x$. Then $\alpha \geq x+k \geq h+k$, and $\alpha \leq y-k < 2k$. This is a contradiction since $h+k \geq 2k$ by hypothesis.

- $y > x > \alpha$. Then $\alpha \leq x-k < k$. However, $y_1 \geq k$ (it is connected by a 2-length path to node 0). Thus $y_1 > \alpha$, which means $y_1 \geq \alpha+h \geq h$. But we know that $y_1 < y$, that is $y_1 \leq y-h < 3k-h$. This is a contradiction since $h \geq 3k-h$ by hypothesis.

Case 2.2: $y_1 > y_2$. By **(1)** above, we have $y_2 < y < y_1$. Let us now look at the relative positions of z, z_1 and z_2. We first note that if $z_m = \min\{z_1, z_2\}$ and $z_M = \max\{z_1, z_2\}$, then $z_m < z_M < z$. Indeed, if $z_M > z$ then $z_M \geq z + h$, and since we know $z \geq h + 2k$, we conclude $z_M \geq h + 3k$, a contradiction.

- $z_1 > z_2$. Hence $z > z_1 > z_2$, by the argument above. Let us derive here some inequalities that will be useful in the following. Since $z < h + 3k$ and $z_1 \leq z - h$, we conclude $z_1 < 3k$. Moreover, we know that $z_2 \geq k$ and $z_1 > z_2$, thus we conclude $z_1 \geq z_2 + k \geq 2k$. Finally, we recall that $h + 2k \leq z < h + 3k$. Now let us look at the relative positions of β and y.
    - $\beta < y$. Then $\beta \leq y - k < 2k$. Since $z_1 \geq 2k$, we conclude $\beta < z_1$. Thus $\beta \leq z_1 - h \leq 3k - h$. We also know that $y_2 \leq 3k - h$ because $y_2 < y \leq y - h$, and because $y < 3k$. Hence, both β and y_2 are contained in the interval $[0; 3k - h[$, of width $w < 3k - h$. But $3k - h \leq h$ by hypothesis, and since β and y_2 must be at least h away, this is impossible.
    - $\beta > y$. Then $\beta \geq y + k \geq h + 2k$. This implies that both β and z lie in the interval $[h + 2k; h + 3k[$, of width $w < k$. However, β and z must be at least k away from each other, a contradiction.
- $z_2 > z_1$. Hence $z > z_2 > z_1$. In particular, we have $k \leq z_1 < 2k$. But we also know that $k \leq y_2 < 3k - h \leq 2k$. Thus y_2 and z_1 both lie in the interval $[k; 2k[$, of width $w < k$. But they must be at least k away, a contradiction.

Altogether, we have shown that every possible case leads to a contradiction. This proves that the initial assumption, $\lambda < h + 3k$, is false. This proves the proposition. □

4 REGULAR GRIDS OF DEGREE 4

4.1 Upper Bounds for G_4

Proposition 7. *$\lambda_{h,k}(G_4) \leq h + 3k$ when $h \leq \frac{k}{2}$.*

Proof. Consider the $L(1,2)$-labeling whose general pattern is depicted in Figure 4.1(a). This labeling has span 7. If we now substitute labels $0, h, k, h + k, 2k, h + 2k, 3k, h + 3k$ to labels $0, 1, \ldots, 7$, the new labeling we obtain is an $L(h,k)$-labeling of G_4. Indeed, it is easy to see that when $h \leq \frac{k}{2}$, each pair of consecutive labels differs by at least h, while each other pair of distinct labels differs by at least k. Moreover, the largest label used is $h + 3k$, hence the result. □

Proposition 8. *$\lambda_{h,k}(G_4) \leq$ min $\{7h, 4k\}$ when $\frac{k}{2} \leq h \leq k$.*

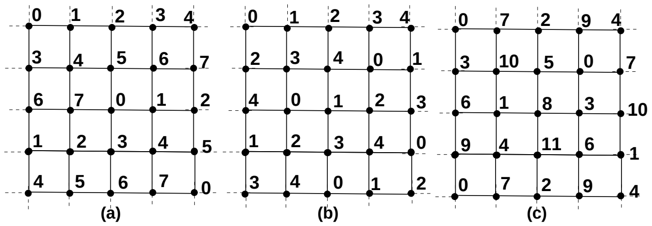


Figure 4 *General patterns for $L(h,k)$-labelings of G_4: (a) $L(1,2)$; (b) $L(1,1)$; (c) $L(3,2)$*

Proof. By Lemma 3, since $\frac{k}{2} \leq h$ and since there exists an $L(1,2)$-labeling of G_4 that is of span 7 (as shown in Figure 4.1(a)), we know there exists an $L(h,k)$-labeling of G_4 of span $7h$.
Analogously, since $h \leq k$, we obtain an $L(h,k)$-labeling of span $4k$ by Lemma 2 ; indeed, there exists an $L(1,1)$-labeling of G_4 that is of span 4 (whose pattern is shown in Figure 4.1(b), see also (1)). □

Proposition 9. *$\lambda_{h,k}(G_4) \leq 3h + k$ when $\frac{3k}{2} \leq h \leq \frac{5k}{3}$.*

Proof. Consider the $L(3,2)$-labeling of G_4 whose general pattern is depicted in Figure 4.1(c). This labeling has span 11. If we now substitute labels $0, h - k, k, h, 2h - k, h + k, 2h, 3h - k, 2h + k, 3h, 4h - k, 3h + k$ to labels $0, 1, \ldots, 11$, the new labeling we obtain is an $L(h,k)$-labeling of G_4. By construction, any pair of labels that are at least 3 away in the list differs by at least h, while any pair of labels that is at least 2 away in the list differs by at least k, because we supposed $\frac{3k}{2} \leq h$. Moreover, the largest label used is $3h + k$, hence the result. □

Proposition 10. *$\lambda_{h,k}(G_4) \leq \frac{11k}{2}$ when $\frac{11k}{8} \leq h \leq \frac{3k}{2}$.*

Proof. It is known (see (5)) that $\lambda_{h,k}(G_4) \leq 4h$ when $h \geq k$. Since $\lambda_{h,k}$ is a non decreasing function, Proposition 9 implies that $\lambda_{h,k}(G_4) \leq \frac{11k}{2}$ when $\frac{11k}{8} \leq h \leq \frac{3k}{2}$. □

4.2 Lower Bounds for G_4

Proposition 11. *$\lambda_{h,k}(G_4) \geq h + 3k$ when $h \leq k$.*

Proof. This bound directly comes from Lemma 1. □

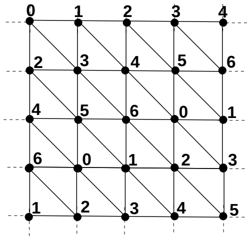
5 REGULAR GRIDS OF DEGREE 6

Proposition 12. *$\lambda_{h,k}(G_6) = 6k$ when $h \leq k$.*

Proof. The upper bound is proved observing that since $h \leq k$, we obtain an $L(h,k)$-labeling of span $6k$ by Lemma 2 ; indeed, there exists an $L(1,1)$-labeling of G_6 of span 6, whose general pattern is shown in Figure 4.2 (see also (1)). The lower bound directly comes from Lemma 1. □

6 REGULAR GRIDS OF DEGREE 8

Figure 5 *General pattern of an $L(1,1)$-labeling of G_6 of span 6*

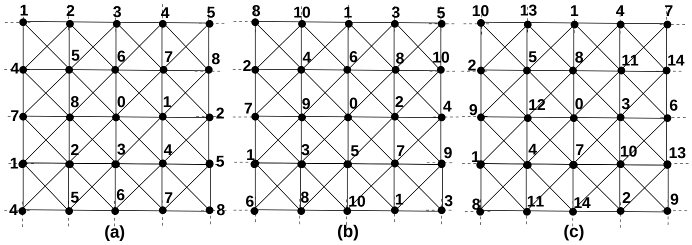


6.1 Upper Bounds for G_8

Proposition 13. *$\lambda_{h,k}(G_8) \leq 8k$ when $h \leq k$.*

Proof. Since $h \leq k$, we obtain an $L(h,k)$-labeling of span $8k$ by Lemma 2 ; indeed, there exists an $L(1,1)$-labeling of G_8 of span 8 (whose general pattern shown in Figure 6.1(a)). ☐

Figure 6 *General patterns for $L(h,k)$-labelings of G_8: (a) $L(1,1)$; (b) $L(2,1)$; (c) $L(3,1)$*



Proposition 14. *$\lambda_{h,k}(G_8) \leq min\ \{8h, 10k\}$ when $k \leq h \leq 2k$.*

Proof. Once again we exploit the $L(1,1)$-labeling of G_8 whose general pattern is depicted in Figure 6.1(a). If we substitute $0, h, 2h, \ldots 8h$ to labels $0, 1, \ldots, 8$, the new labeling we obtain is an $L(h,k)$-labeling of G_8. Indeed, it is easy to see that each pair of consecutive labels differs by at least h, and thus by at least k since $k \leq h$. Moreover, the largest label used is $8h$, hence the result.

The upper bound of $10k$ comes from the $L(2,1)$-labeling of G_8 whose general pattern is shown in Figure 6.1(b). If we substitute $0, k, 2k, \ldots 10k$ to labels $0, 1, \ldots, 10$, the new labeling we obtain is an $L(h,k)$-labeling of G_8. Indeed, it is easy to see that when $k \leq h \leq 2k$, each pair of non consecutive labels differs by at least $2k \geq h$, while any pair of distinct labels differs by at least k. Moreover, the largest label used is $10k$, hence the result. ☐

Proposition 15. *$\lambda_{h,k}(G_8) \leq min\ \{5h, 14k\}$ when $2k \leq h \leq 3k$.*

Proof. Consider the $L(2,1)$-labeling whose general pattern is described in Figure 6.1(b). This labeling has span 10. If we now substitute $0, k, h, h+k, 2h, 2h+k, 3h, 3h+k, 4h, 4h+k, 5h$ to labels $0, 1, \ldots, 10$, the new labeling we obtain is an $L(h,k)$-labeling of G_8. Indeed, it is easy to see that each pair of non consecutive labels differs by at least h. On the other hand, since $2k \leq h$, any pair of distinct labels differs by at least k. Moreover, the largest label used is $5h$.

Analogously, the other bound is given using an $L(3,1)$-labeling, such as the one whose general pattern is shown in Figure 6.1(c). This labeling is of span 14. If we now substitute $0, k, 2k, \ldots, 14k$ to labels $0, 1, \ldots, 14$, the new labeling we obtain is an $L(h,k)$-labeling of G_8. Indeed, when $h \leq 3k$, each pair of labels that are at least 3 away in the list differs by at least $3k \geq h$, while any pair of distinct labels differs by at least k. Moreover, the largest label used is $14k$, hence the result. ☐

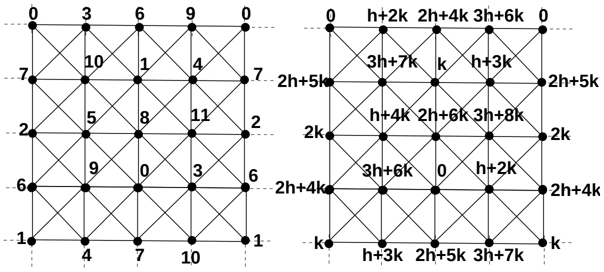
Proposition 16. *$\lambda_{h,k}(G_8) \leq 4h + 2k$ when $3k \leq h \leq 6k$.*

Proof. Starting from the $L(3,1)$-labeling used in the previous proof (cf. also Figure 6.1(c)) of span 14, we substitute labels $0, k, 2k, h, h, h+k, h+2k, 2h, 2h+k, \ldots, 4h, 4h+k, 4h+2k$ to labels $0, 1, \ldots, 14$. This new labeling is also an $L(h,k)$-labeling of G_8. Indeed, each pair of labels that are at least 3 away in the list differs by at least h by construction, while any pair of distinct labels differs by at least k because $h \geq 3k$. Moreover, the largest label used is $4h+2k$, hence the result. ☐

Proposition 17. *$\lambda_{h,k}(G_8) \leq 3h + 8k$ when $h \geq 6k$.*

Proof. Consider the labeling whose general pattern is depicted in Figure 6.1(a). This labeling is an $L(1,1)$-labeling of span 11, with the additional property that the only consecutive labels that can appear on neighboring nodes are of the form $3i+2$ and $3(i+1)$. We now replace any label l of this labeling by a new label, thanks to the following rule (cf. Figure6.1(b)): any label of the form $l = 3i + j$ ($i = 0, 1, 2, 3$, $j = 0, 1, 2$) is replaced by $l' = (h+2k)i + jk$. In this new labeling, any pair of labels of the form $3i+2$ and $3(i+1)$ is now separated by h. Moreover, the labeling we started from is an $L(1,1)$-labeling, and any two differing labels in the new labeling are at least k away. Thus, this new labeling is an $L(h,k)$-labeling, of span $3h + 8k$. ☐

Figure 7 *(a) General pattern of an $L(1,1)$-labeling of G_8 ; (b) general pattern of the $L(h,k)$-labeling we derive*



6.2 Lower Bounds for G_8

Proposition 18. *$\lambda_{h,k}(G_8) \geq 8k$ when $h \leq k$.*

Proof. This bound directly comes from Lemma 1. □

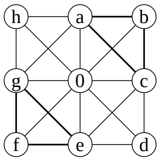
Proposition 19. *$\lambda_{h,k}(G_8) \geq 2h + 6k$ when $k \leq h \leq 3k$.*

Proof. Consider any optimal $L(h,k)$-labeling of G_8. Let λ be the greatest label. Let us consider a label x which is neither 0 nor λ (note that there must exist one since G_8 contains K_3 as an induced subgraph ; note also that necessarily, x lies in the interval $[h; \lambda - h]$). Now, consider its 8 neighbors, say $v_1 \dots v_8$. Then no other label than x can be used in the interval $]x-h; x+h[$ for the v_is. However, all the v_is are pairwise connected by 2-length paths, so they must be at least k away from each other. If there are α (resp. β) labels for the v_is in the interval $[0; x-h]$ (resp. $[x+h; \lambda]$), then we must have $(x-h) - (\alpha - 1)k \geq 0$ and $\lambda \geq (x+h) + (\beta - 1)k$, with $\alpha + \beta = 8$. Since $\lambda_{h,k}(G_8) = \lambda$, we conclude that $\lambda_{h,k}(G_8) \geq 2h + (\alpha + \beta - 2)k$, hence the result. □

Proposition 20. *$\lambda_{h,k}(G_8) \geq 3h + 3k$ when $h \geq 3k$.*

Proof. First, observe that we have $\lambda_{h,k}(G_8) \geq 3h + k$. Indeed, consider an optimal $L(h,k)$-labeling of G_8, a node labeled 0, and the set of its neighbors (see Figure 6.2). Wlog, suppose $\min\{a,b,c\} \leq \min\{e,f,g\}$. Since a, b and c are neighbors of 0, then we have $\min\{a,b,c\} \geq h$. And since any node among e, f and g are connected by a 2-length path to any node among a, b and c, we conclude that $\min\{e,f,g\} \geq h + k$. Finally, since e, f and g induce a K_3, we have $\max\{e,f,g\} \geq 3h + k$.

Figure 8 *Neighborhood of a node labeled 0 in G_8.*



However, we can derive a better lower bound of $3h + 3k$, taking into account nodes d and h in addition to the previous study. This bound then derives from a very tedious case by case analysis that is not developed here. Instead, we have run an exhaustive search by computer on the grid restricted to those nine nodes. The source and binary codes corresponding to this search are available at the following URL: `http://www.sciences.univ-nantes.fr/info/perso/permanents/fertin/Lhk/Lhk.c`). □

7 CONCLUDING REMARKS

In this paper, we have studied the $L(h,k)$-labeling problem on regular grids of degree 3, 4, 6 and 8, and we have improved, in many different cases, the bounds on the $L(h,k)$ number in each of these classes of graphs. A graphical representation of our results is depicted in Figure 6.2: bold lines in this figure are results from this paper, grey lines are previously known results, and grey zones represent the gaps that still exist between the known lower and upper bounds.

Though we managed to obtain tight bounds in several cases, there are still some other cases for which the gap is not closed, and it actually looks difficult to improve the bounds without using case by case analysis arguments, as we have sometimes done in this paper. However, a natural question consists in closing the gaps that still remain in all the four classes of graphs considered here.

Moreover, as observed in the introduction, when $h < k$ we have considered in this paper the *max-based* model, that imposes a condition on labels of nodes connected by a *2-length path* instead of using the concept of *distance 2* (we recall that when $h \geq k$, the two definitions coincide). Hence, it is also natural to ask for a similar study in the case $h < k$, but using this time the *distance-based* definition. We note that this makes sense only for G_6 and G_8, since there are no triangles in G_3 and G_4, and thus in that case the two definitions coincide. Moreover, since the *max-based* model is by definition more restrictive than the *distance-based* model, the upper bounds we obtain in the *max-based* model also apply in the *distance-based* model, while this is not a priori the case for lower bounds.

REFERENCES

[1] R. Battiti, A.A. Bertossi and M.A. Bonuccelli. Assigning Codes in Wireless Networks: Bounds and Scaling Properties. *Wireless Networks*, **5**:195–209, 1999.

[2] A.A. Bertossi and M.A. Bonuccelli. Code Assignment for Hidden Terminal Interference Avoidance in Multihop Packet Radio Networks. *IEEE/ACM Trans. on Networking*, **3**:441–449, 1995.

[3] A.A. Bertossi, C.M. Pinotti and R.B. Tan. Channel assignment with separation for interference avoidance in wireless networks. *IEEE Transactions on Parallel and Distributed Systems* **14**:222–235, 2003.

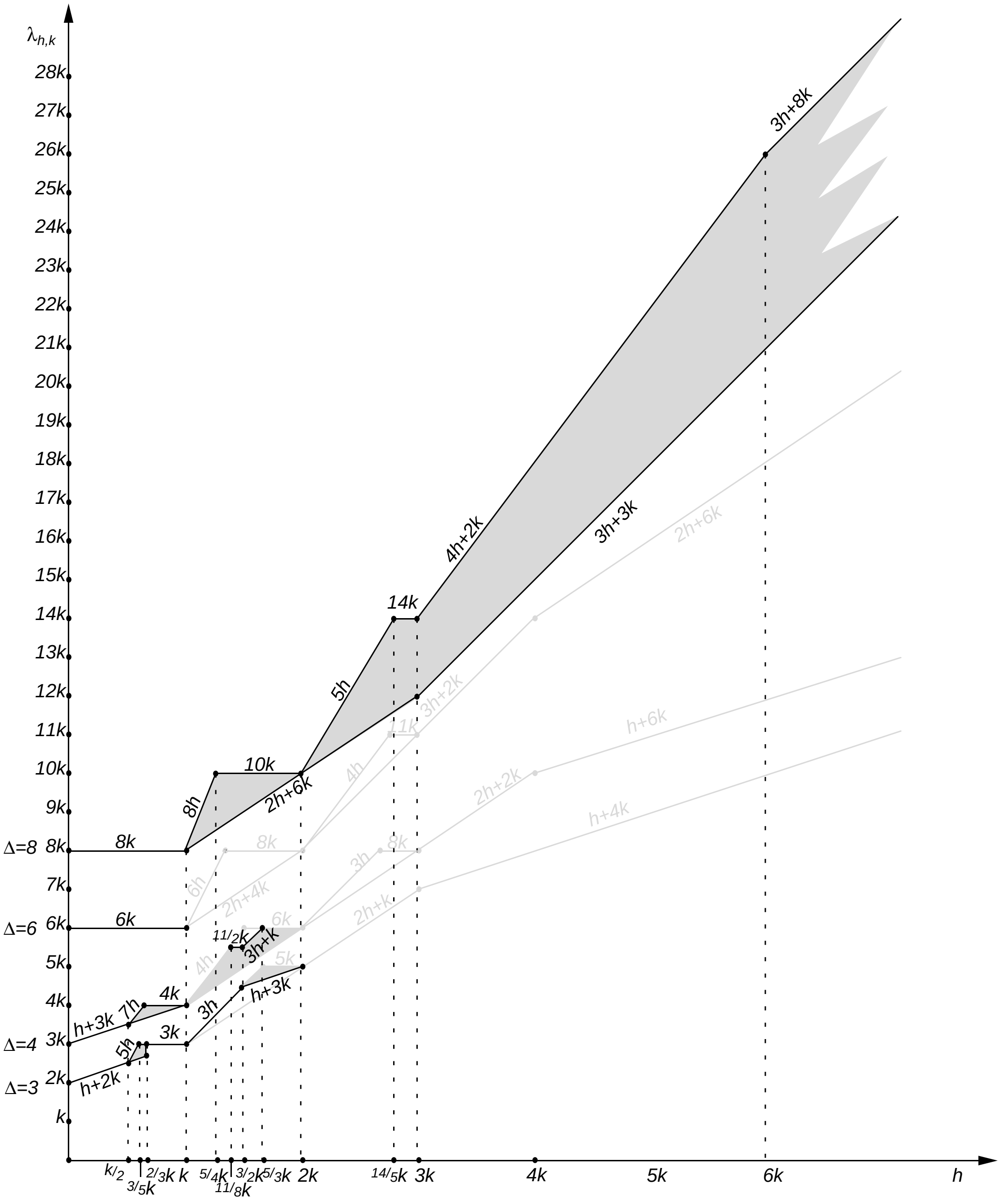
[4] H.L. Bodlaender, T. Kloks, R.B. Tan and J. van Leeuwen. λ-Coloring of Graphs. In *Proc. of STACS 2000*. LNCS 1770, 395–406, 2000.

[5] T. Calamoneri. Exact Solution of a Class of Frequency Assignment Problems in Cellular Networks and Other Regular Grids. *Proc. 8th Italian Conference on Theoretical Computer Science* (ICTCS'03), LNCS 2841, 163–173, 2003.

[6] T. Calamoneri. The $L(h,k)$-Labelling Problem: A Survey. Tech. Rep. 04/2004 Univ. of Rome "La Sapienza", Dept. of Computer Science, 2004.

[7] T. Calamoneri and R. Petreschi. $L(h,1)$-Labeling Subclasses of Planar Graphs. *Journal on Parallel and Distributed Computing* **64(3)**: 414-426, 2004.

Figure 9 *Summary of the results achieved in this paper: bold lines are results from this paper, grey lines are previously known results, and grey zones represent the gaps that still exist between the known lower and upper bounds.*



[8] G.J. Chang, W.-T. Ke, D. Kuo, D.D.-F. Liu and R.K. Yeh. On $L(d,1)$-labelings of graphs. *Discrete Mathematics* **220**:57–66, 2002.

[9] G.J. Chang and D. Kuo. The $L(2,1)$-labeling Problem on Graphs. *SIAM J. Disc. Math.*, **9**:309–316, 1996.

[10] G. Fertin and A. Raspaud. $L(p,q)$ Labeling of d-Dimensional Grids. *Discrete Applied Mathematics*, to appear.

[11] J.R. Griggs and R.K. Yeh. Labeling graphs with a Condition at Distance 2. *SIAM J. Disc. Math*, **5**:586–595, 1992.

[12] W.K. Hale. Frequency assignment: theory and applications. *Proc. IEEE*, **68**:1497–1514, 1980.

[13] P.K. Jha, A. Narayanan, P. Sood, K. Sundaram and V. Sunder. On $L(2,1)$-labeling of the Cartesian product of a cycle and a path. *Ars Combin.* **55**:81–89, 2000.

[14] P.K. Jha. Optimal $L(2,1)$-labeling of Cartesian products of cycles with an application to independent domination. *IEEE Trans. Circuits & Systems I: Fundamental Theory and Appl.* **47**:1531–1534, 2000.

[15] D. Korže and A. Vesel. L(2,1)-labeling of strong products of cycles. Information Processing Letters **94**:183-190, 2005.

[16] T. Makansi. Transmitter-Oriented Code Assignment for Multihop Packet Radio. *IEEE Trans. on Comm.*, **35(12)**:1379–1382, 1987.

[17] B.H. Metzger. Spectrum Management Technique. *In Proc. 38th National ORSA Meeting*, LNCS 1770, 395–406, 1970.

[18] M. Molloy and M.R. Salavatipour. Frequency channel assignment on planar networks. In Proceedings of *10th Annual European Symposium on Algorithms (ESA)*, LNCS 2461, 736–747, 2002.

[19] D. Sakai. Labeling Chordal Graphs: Distance Two Condition. *SIAM J. Disc. Math*, **7**:133–140, 1994.

[20] M.A. Whittlesey, J.P. Georges and D.W. Mauro. On the λ number of Q_n and related graphs, *SIAM J. Discr. Math.* **8**:499-506, 1995.