

Hierarchical Broadcast and Gossip Networks

Guillaume Fertin

LaBRI U.M.R. C.N.R.S. 5800, Université Bordeaux I
351 Cours de la Libération, F33405 Talence Cedex
fertin@labri.u-bordeaux.fr

Abstract

In this paper, we study hierarchical broadcast (resp. gossip) networks. These networks are such that every connected induced subgraph is also a broadcast (resp. gossip) network. This study follows and generalizes the study of hierarchical broadcast networks in the undirected case (cf. [7]). Here, we study broadcasting in the directed case, while gossiping is studied in the unit cost as well as the linear cost model. We prove that, for any of these three models, the minimum outdegree (resp. degree) of a directed hierarchical broadcast network (resp. hierarchical gossip network, hierarchical linear gossip network) of order n is equal to $n - 2$.

Keywords : broadcasting, gossiping, networks.

1 Introduction

Broadcasting and gossiping are two information dissemination problems. In *broadcasting*, one node of a network possesses a piece of information, and transmits it to every other node. In *gossiping*, all the nodes start with a piece of information, and, at the end of the process, every node in the network must know the information of every other node. Communications are performed by calls between pairs of nodes, along the links of the network. We consider the *1-port store-and-forward* model, in which each call involves two nodes, and the single communication link connecting them. Moreover, each node communicates with at most one of its neighbours at any given time. We assume that each node starts with a message of length 1, and that the set of informations known by a node can be concatenated to form a single message.

First, we recall the results concerning broadcasting in the undirected case. Let b_n be the minimum time to broadcast in the complete network, i.e. a network which possesses all the possible links. It is well-known that $b_n = \lceil \log_2(n) \rceil$ for any n . A network able to achieve broadcasting in time b_n is then called a *broadcast network*. A *hierarchical broadcast network* of order n , or HBN_n , denotes a broadcast network such that every connected induced subgraph is also a broadcast network. Let us denote by $B(n)$ the minimum number of edges of a hierarchical broadcast network of order n . We have the following result from [7].

Theorem 1 ([7]) $B(1) = 0$, $B(2) = 1$, $B(3) = 2$ and for any $n \geq 4$, $B(n) = \lceil \frac{n(n-2)}{2} \rceil$.

In this paper, we extend and generalize this result to three other models of communication. We first consider broadcasting in the directed case. The graph modelling the network is therefore directed. We also consider gossiping in the undirected case under two models : the *unit cost* model and the *linear cost model*. In the former model, a communication takes one time unit, independently of the amount of information transmitted. In the latter model, the time to send a message of length L (that is, a message composed with L pieces of information) is $\beta + L\tau$, where β is the *start-up* time to initiate a call between a pair of nodes, and τ is the *propagation* time of a message of length 1 along a link. We refer to [8, 10, 11] for more informations about broadcasting

and gossiping.

For each of the three models described above, we use similar definitions as in broadcasting in undirected networks. More precisely, we have the following :

- The minimum time \vec{b}_n to achieve broadcasting in the fully-connected directed network with n nodes satisfies $\vec{b}_n = \lceil \log_2(n) \rceil$. The minimum number of arcs of a directed hierarchical broadcast network of order n (also called $DHBN_n$) is denoted by $\vec{B}(n)$;
- The minimum time g_n to achieve gossiping under the *unit cost* model in the fully-connected network with n nodes satisfies $g_n = \lceil \log_2(n) \rceil + \text{odd}(n)$, where $\text{odd}(n) = 1$ if n is odd, and 0 otherwise [12]. The minimum number of edges of a hierarchical gossip network of order n (also called HGN_n) is denoted by $G(n)$;
- The minimum time $g_{\beta,\tau}(n)$ to achieve gossiping under the *linear cost* model in the fully-connected network with n nodes satisfies $g_{\beta,\tau}(n) = \lceil \log_2(n) \rceil \beta + (n-1)\tau$ for any even n [9]. Note that in this model, very little is known when n is odd : in particular, the optimal time to achieve gossiping in the linear cost model in the fully connected n -node network when n is odd is not precisely known, and its study turns out to be complicated [4, 5, 6]. This is why we will focus here on the case n even only (note that for the same reasons, cf. [1, 2], we will not consider in this paper gossiping in the 1-way mode, as well as gossiping in directed networks).

In the linear cost model, when we talk about hierarchical gossip networks, this means gossip networks such that any induced connected subgraph of *even order* is a gossip graph. The minimum number of edges of a hierarchical linear gossip network of order n (also called $HLGN_n$) is then denoted by $G_{\beta,\tau}(n)$.

From an application perspective, these networks represent the cheapest architecture (in terms of communication links) that keeps efficient communication capabilities in each of its (connected) parts.

In this paper, we give the optimal values of $\vec{B}(n)$ and $G(n)$ for any n , and of $G_{\beta,\tau}(n)$ for any even n . In Section 2, we study directed hierarchical broadcast networks, and we give the exact value of $\vec{B}(n)$ for all n . In Section 3, we consider hierarchical gossip networks in the undirected case. We first assume a *unit cost* model in Section 3.1, and we determine the exact size $G(n)$ of a hierarchical gossip network for all n . Then, in Section 3.2 we study the *linear cost* model. In this model, we determine the size of linear gossip networks of *even order* for which every induced connected subgraph of *even order* is also a linear gossip network.

2 Directed Hierarchical Broadcast Networks

Thanks to Theorem 1, we can derive a result for *directed* hierarchical broadcast networks. First, one can check that $\vec{B}(1) = 0$, $\vec{B}(2) = 2$, $\vec{B}(3) = 4$ and $\vec{B}(4) = 8$, thanks to the results of [13] (cf. Figure 1).

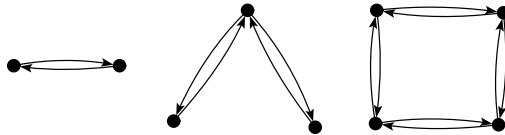


Figure 1: From left to right : a $DHBN_2$, a $DHBN_3$ and a $DHBN_4$

Lemma 1 For all $n \geq 4$, $\vec{B}(n) \geq n(n-2)$.

Proof : The lower bound proof relies on the same arguments as in proof of Theorem 1. We show by induction that any vertex v of a $DHBN_n$ has an outdegree $d^+(v) \geq n - 2$, and an indegree $d^-(v) \geq n - 2$. This is true for $n = 4$. Now suppose that this property holds for a fixed $n \geq 4$, and let us show that it holds for $n + 1$. Let us consider a $DHBN_{n+1}$, say G_{n+1} . Let us consider an arbitrary vertex u in G_{n+1} . Let $G_1, G_2 \dots G_k$ be the connected components of $G_{n+1} - \{u\}$, $1 \leq k \leq n$. Let $n_i = |V(G_i)|$ for any $i = 1, \dots, k$. W.l.o.g., we suppose $n_1 \geq n_2 \geq \dots \geq n_k$. Let v be a vertex in G_1 such that the (undirected) distance $d(u, v)$ between u and v is maximum.

$G_{n+1} - \{v\}$ is connected because every $G_i, i \geq 2$, is connected to u , and if the deletion of v would disconnect G_1 , then v would not be such that $d(u, v)$ is maximum. Since $G_{n+1} - \{v\}$ is connected, it is a $DHBN_n$, by definition. Hence, all vertices of $G_{n+1} - \{v\}$ have indegree and outdegree at least $n - 2$ by induction hypothesis.

Moreover, $n_1 \geq 2$. Otherwise, the broadcast process from u would take strictly more than $\lceil \log_2(n+1) \rceil$ rounds in G_{n+1} . Hence, there exists a vertex u_1 in G_1 such that $u_1 \neq v$. Thus $d^+(u_1) \geq n - 2$ and $d^-(u_1) \geq n - 2$ in $G_{n+1} - \{v\}$. Hence $n_1 \geq n - 1$. However, $n_1 = n - 1$ is impossible. Indeed, if $n_1 = n - 1$, then let $w \neq u$ be the vertex of G_{n+1} not in G_1 . Necessarily, we have $n + 1 = 2^{p-1} + 1$ for a given p , otherwise w would not be able to broadcast in G_{n+1} in $\lceil \log_2(n+1) \rceil$ rounds. Moreover, we must have an arc (u, w) , and an arc (w, u) in G_{n+1} , otherwise broadcasting either from or to w in G_{n+1} would be impossible. Now let a vertex $u' \in G_1$ be a neighbour of u . Clearly, $G_{n+1} - \{u'\}$ is connected. By hypothesis, it has to be a broadcast digraph. However, it holds 2^{p-1} vertices, and w cannot broadcast its information to all the other vertices in the graph in $p - 1$ rounds. This proves that $n_1 = n$ and thus $k = 1$.

To resume, we have that every vertex of $G_1 = G_{n+1} - \{u\}$ is of indegree (resp. outdegree) at least $n - 2$, because G_1 is connected of order n . Moreover, $G_{n+1} - \{v\}$ is connected too, hence $d^+(u) \geq n - 2$ and $d^-(u) \geq n - 2$ in $G_{n+1} - \{v\}$. This means that all the out-neighbours of u in $G_{n+1} - \{v\}$ (there are at least $n - 2$ such vertices) have indegree at least $n - 1$, and all the in-neighbours of u in $G_{n+1} - \{v\}$ have outdegree at least $n - 1$. Since u has been chosen arbitrarily, we conclude that every vertex in G_{n+1} has outdegree and indegree at least equal to $n - 1$, which shows by induction that $\vec{B}(n) \geq n(n - 2)$ for all $n \geq 4$. \square

Now let us show that $\vec{B}(n) \geq n(n - 2) + 1$ when n is odd. For this, we show the following lemma.

Lemma 2 *For any odd $n \geq 5$, any $DHBN_n$ has at least one vertex w such that $d^+(w) \geq n - 1$ or $d^-(w) \geq n - 1$. Thus $\vec{B}(n) \geq n(n - 2) + 1$.*

Proof : In any $DHBN_n, n \geq 4$, all the vertices have outdegree and indegree at least $n - 2$ (cf. Lemma 1). Now, suppose n is odd and suppose there exists a $DHBN_n G$ such that every vertex u of G satisfies $d^+(u) = d^-(u) = n - 2$. Then, let us consider a vertex u_1 of G . Since $d^+(u_1) = n - 2$, there exists a vertex v_1 of G such that $(u_1, v_1) \notin A(G)$, where $A(G)$ is the set of arcs of G . Moreover, $(v_1, u_1) \notin A(G)$, because otherwise the graph induced by vertices u_1 and v_1 would be connected and therefore would be a broadcast digraph. That is, we would have $(v_1, u_1) \in A(G)$ and $(u_1, v_1) \in A(G)$. Hence, we have the following property : for every vertex u_i in G , there exists a unique vertex v_i in G such that u_i and v_i are not neighbours (Property **P1**). Hence, we can consider $\frac{n-1}{2}$ such pairs $\{u_i, v_i\}$, where each vertex appears only once. Since n is odd, there remains a vertex w in G for which **P1** does not hold. That is, for all $v \neq w \in V(G)$, there exists an arc (v, w) or (exclusively) an arc (w, v) . This means that w is neighbour of all the $n - 1$ other vertices in G . Thus, for any vertex $v \neq w$, the graph induced by vertices v and w is connected, and must then be a broadcast digraph. That is, for any $v \neq w, (v, w) \in A(G)$ and $(w, v) \in A(G)$. Hence $d^+(w) = d^-(w) = n - 1$. Consequently, when n is odd, there exists at least one vertex w in any $DHBN_n$ such that $d^+(w) \geq n - 1$ or $d^-(w) \geq n - 1$. \square

Theorem 2 (Directed Hierarchical Broadcast Networks) $\vec{B}(1) = 0, \vec{B}(2) = 2, \vec{B}(3) = 4$ and for any $n \geq 4, \vec{B}(n) = n(n - 2) + \text{odd}(n)$, where $\text{odd}(n) = 1$ if n is odd, and 0 otherwise.

Proof : Thanks to the previous lemmas, $\vec{\mathbf{B}}(n) \geq n(n-2) + \text{odd}(n)$ for any $n \geq 4$. Moreover, Fraigniaud has shown that $\mathbf{B}(n) = \lceil \frac{n(n-2)}{2} \rceil$ in the undirected case (cf. Theorem 1). Therefore, by replacing every edge of an HBN_n by a pair of symmetric arcs, we get a $DHBN_n$ (thanks to the fact that $b_n = \vec{b}_n = \lceil \log_2(n) \rceil$ for any n). Hence $\vec{\mathbf{B}}(n) \leq 2 \cdot \mathbf{B}(n)$ for any n , and the result follows. \square

3 Hierarchical Gossip Networks

3.1 The unit cost model

In [7], Fraigniaud studied a subclass of hierarchical gossip networks (in the *unit cost* model). These networks are of *even* order and are such that any of their connected induced subgraphs of *even* order are gossip networks. The difficulty here of gossiping in the *unit cost* model is that the time g_{2m} to gossip in the complete graph K_{2m} satisfies $g_{2m} = \lceil \log_2(2m) \rceil$, while the time g_{2m+1} to gossip in K_{2m+1} satisfies $g_{2m+1} = \lceil \log_2(2m+1) \rceil + 1$. However, it is actually possible to consider both the even and odd cases, and therefore to generalize Corollary 1 from [7] to all n .

Theorem 3 (Hierarchical Gossip Networks) $\mathbf{G}(1) = 0$, $\mathbf{G}(2) = 1$, $\mathbf{G}(3) = 2$ and for all $n \geq 4$, $\mathbf{G}(n) = \lceil \frac{n(n-2)}{2} \rceil$.

Proof : The lower bound on $\mathbf{G}(n)$ relies on the same proof as the proof of Theorem 1 in [7]. When $n \leq 4$, the lower bound can be shown to be true by inspection. As in [7], we can show the following property by induction : for any $n \geq 4$, every vertex in an HGN_n has a degree at least equal to $n-2$. The proof is the same as the proof of Theorem 1 in [7] (and very similar to the proof of Lemma 1). Those proofs do not rely neither on the type of communication (broadcasting or gossiping) nor on the minimum time to achieve it, except for two arguments, that we detail below (we refer to proof of lemma 1 to figure out where these differences take place).

- $n_1 \geq 2$, since otherwise G_{n+1} would be the star of order $n+1$, of center u , and with n pendent edges. However, it has been shown in [3] that any gossip graph of order $n+1$ has at most $\lfloor \frac{n+1}{2} \rfloor$ pendent edges, a contradiction.
- $n_1 = n$. Indeed, suppose $n_1 = n-1$, and let $w \neq u$ be the only vertex not in G_1 . In that case, the edge uw is a pendent edge, and necessarily $n+1$ is odd (because there is no vertex of degree 1 in a gossip graph of even order $n \geq 4$, cf. [3]). Now let u' be a neighbour of u in G_1 . $G_{n+1} - \{u'\}$ is connected, and thus it is an HGN_n , with n even. However, $G_{n+1} - \{u'\}$ has at least one pendent edge uw , and we know from [3] that this is impossible (a gossip graph of even order $n \geq 4$ has minimum degree at least 2). Hence $n_1 = n$.

The upper bound comes from [7], where a family of graphs G_n is constructed recursively as follows : we start from $G_4 = C_4$. G_{2k+1} ($k \geq 2$) is obtained from G_{2k} by adding one vertex connected to all the vertices of G_{2k} , while G_{2k+2} is obtained from G_{2k} by adding two vertices, both connected to all the vertices of G_{2k} . Now, let us prove that for any $n \geq 4$, any graph G_n of the family constructed in [7] is a gossip graph. This will prove that G_n is an HGN_n for any $n \geq 4$. For this, we have to distinguish two cases :

- n is even. Let us show that the graph G_n given by construction contains the Knödel graph $W_{n, \lceil \log_2(n) \rceil}$ [12]. Since $W_{n, \lceil \log_2(n) \rceil}$ is a gossip graph for any even n [12], this will imply that G_n is a gossip graph. Note that the proof given in [7] for the upper bound for even n is not formally correct, though the result is correct. This is why we revisit the case n even. Every vertex in G_n is of degree $n-2$. Hence, for any vertex u in G_n , there exists a unique vertex v such that $uv \notin E(G_n)$. That is, we can partition $V(G_n)$ into $\frac{n}{2}$ couples of vertices $\{u_i, v_i\}$ such that $u_i v_i \notin E(G_n)$ for any $1 \leq i \leq \frac{n}{2}$. We now distinguish two more cases :

- $n = 4m$. We partition $V(G_n)$ into two sets, V_1 and V_2 , such that $V_1 = \{\{u_i, v_i\}, 1 \leq i \leq m\}$ and $V_2 = \{\{u_i, v_i\}, m+1 \leq i \leq 2m\}$ (cf. Figure 2). We have $|V_1| = |V_2|$, and any vertex of V_1 is connected to all the vertices of V_2 . Hence, the complete bipartite graph $K_{2m, 2m}$ is a subgraph of G_n , which we note $K_{2m, 2m} \subseteq G_n$. Since $W_{n, \lfloor \log_2(n) \rfloor} \subseteq K_{\frac{n}{2}, \frac{n}{2}}$, we conclude that G_n is a gossip graph.

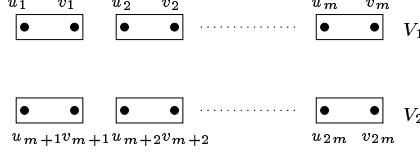


Figure 2: $K_{2m, 2m}$ is subgraph of G_{4m}

- $n = 4m + 2$. In that case, we cannot split $V(G_n)$ into V_1 and V_2 as before. Let $V_1 = \{\{u_i, v_i\}, 1 \leq i \leq m\} \cup \{u_{m+1}\}$ and $V_2 = \{\{u_i, v_i\}, m+2 \leq i \leq 2m+1\} \cup \{v_{m+1}\}$. We have $|V_1| = |V_2|$. Let $K_{2m+1, 2m+1}^*$ be the complete bipartite graph to which we have removed one edge. We can see that $K_{2m+1, 2m+1}^* \subseteq G_n$, where the edge which has been deleted is $u_{m+1}v_{m+1}$. But we can arrange the vertices such that this “missing” edge is an edge which does not exist in the Knödel graph $W_{n, \lfloor \log_2(n) \rfloor}$ (cf. Figure 3). Indeed, $W_{n, \lfloor \log_2(n) \rfloor}$ is $\lfloor \log_2(n) \rfloor$ -regular, and every vertex of $K_{2m+1, 2m+1}^*$ has a degree at least equal to $2m = \frac{n-2}{2}$. Therefore $W_{n, \lfloor \log_2(n) \rfloor} \subseteq K_{2m+1, 2m+1}^*$, and G_n is a gossip graph.

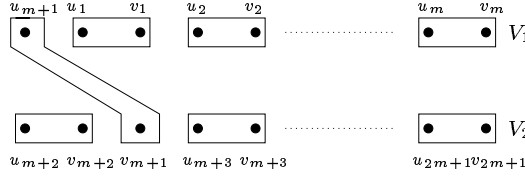


Figure 3: $K_{2m, 2m}^*$ is subgraph of G_{4m+2}

- n is odd. Let p be the integer such that $2^p < n < 2^{p+1}$. We have $\lfloor \log_2(n) \rfloor + 1 = p + 2$. Consider G_{2^p} , and let $V(G_{2^p}) = \{u_i, 1 \leq i \leq 2^p\}$. We have seen in the case n even that G_{2^p} is a gossip graph, and thus gossiping can be achieved in G_{2^p} in p rounds. We now construct a gossip graph of order n from G_{2^p} as follows. Let $n' = n - 2^p$. We have $n' < 2^p$. Let us consider n' distinct vertices $u_j, j = 1, \dots, n'$ in G_{2^p} . We then add to G_{2^p} n' vertices $v_j, j = 1, \dots, n'$, and n' pendent edges $u_j v_j$. This new graph is clearly a subgraph of G_n . Since it is possible to gossip in this graph in $p + 2$ rounds (gossip along the pendent edges during rounds 1 and $p + 2$, while gossiping in G_{2^p} takes place between rounds 2 and $p + 1$), we see that for any odd n , G_n is a gossip graph.

□

3.2 The linear cost model

In the linear cost model, very little is known when the number of vertices n is odd. Hence, we recall here that we will only focus on the case n even. Thus, a *hierarchical gossip network* denotes here a gossip network such that any induced connected subgraph of *even order* is a gossip graph.

Theorem 4 (Hierarchical Linear Gossip Networks of even order) *Let $G_{\beta, \tau}(n)$ be the number of edges of a hierarchical gossip network of even order n in the linear cost model. For any $\beta > 0$, $G_{\beta, \tau}(2) = 1$, and $G_{\beta, \tau}(n) = \frac{n(n-2)}{2}$ for any even $n \geq 4$.*

Proof : It has been shown in [9] that the Knödel graph $W_{n, \lfloor \log_2(n) \rfloor}$ is a linear gossip graph for any even n . Hence, the upper bound proof of Theorem 3 concerning the unit cost model remains valid in the linear cost model, and we have $G_{\beta, \tau}(n) \leq \frac{n(n-2)}{2}$ for any even $n \geq 4$.

Moreover, for any $\beta > 0$, gossiping in the linear cost model implies gossiping in the unit cost model. More precisely, the unit cost model is a special case of the linear cost model, where we take $\beta = 1$ and $\tau = 0$. Indeed, if $\tau = 0$, then only the number of rounds is taken into account, and it follows that $G(n) = G_{1,0}(n)$ for any n (cf. [9]).

Thus, for any even $n \geq 4$, we have $G_{\beta, \tau}(n) \geq G(n)$, that is $G_{\beta, \tau}(n) \geq \frac{n(n-2)}{2}$. \square

4 Conclusion

The results given here generalize and complete the ones of Fraigniaud [7]. Here, we also consider broadcasting in the directed case, and gossiping in the *unit cost* and *linear cost* models.

A summary of all the results displayed in this paper is given in Tables 1 (broadcasting) and 2 (gossiping).

Broadcasting		
	Undirected	Directed
even n	$B(n) = \frac{n(n-2)}{2}$ [7]	$\vec{B}(n) = n(n-2)$ (Th. 2)
odd n	$B(n) = \frac{n(n-2)+1}{2}$ [7]	$\vec{B}(n) = n(n-2) + 1$ (Th. 2)

Table 1: Summary of the results : Hierarchical Broadcast Networks

Gossiping		
	Unit Cost Model	Linear Cost Model
even n	$G(n) = \frac{n(n-2)}{2}$ (Th. 3)	$G_{\beta, \tau}(n) = \frac{n(n-2)}{2}$ (Th. 4)
odd n	$G(n) = \frac{n(n-2)+1}{2}$ (Th. 3)	

Table 2: Summary of the results : Hierarchical Gossip Networks

In each of these cases, we see that directed hierarchical broadcast networks (resp. hierarchical gossip networks, hierarchical linear gossip networks) are very dense ; this tends to show that the hierarchical property is very costly in terms of communication links, at least concerning communication patterns such as broadcasting and gossiping.

References

- [1] R.C. Entringer and P.J. Slater. Gossips and telegraphs. *J. Franklin Institute*, 307:353–360, 1979.

- [2] S. Even and B. Monien. On the number of rounds necessary to disseminate information. *Proc. 1st ACM Symp. on Parallel Algo. and Architectures*, pages 318–327, June 1989.
- [3] G. Fertin. A study of minimum gossip graphs. *Discrete Mathematics*, 1999. To appear.
- [4] G. Fertin. Trade-offs for odd gossiping. In *Proc. 6th International Colloquium on Structural Information and Communication Complexity (SIROCCO '99)*, volume 5 of *Proceedings in Informatics*, pages 137–151. Carleton University Press, 1999.
- [5] G. Fertin and J.G. Peters. Odd gossiping in the linear cost model. *Proc. Workshop on Communication - 23rd Int. Symp. on Mathematical Foundations of Computer Science (MFCS), Brno, Czech Republic*, 1998.
- [6] G. Fertin and J.G. Peters. Optimal odd gossiping. Technical Report CMPT TR 1998-24, Simon Fraser University, Burnaby, B.C., December 1998. Available at <ftp://fas.sfu.ca/pub/cs/TR/1998/CMPT1998-24.ps.gz>. Submitted for publication.
- [7] P. Fraigniaud. Hierarchical broadcast networks. *Information Processing Letters*, 68:303–305, 1998.
- [8] P. Fraigniaud and E. Lazard. Methods and problems of communication in usual networks. *Discrete Applied Math.*, 53:79–133, 1994.
- [9] P. Fraigniaud and J.G. Peters. Minimum linear gossip graphs and maximal linear (Δ, k) -gossip graphs. Technical Report CMPT TR 94-06, Simon Fraser University, Burnaby, B.C., 1994. Available at <http://fas.sfu.ca/pub/cs/techreports/1994/>.
- [10] S.M. Hedetniemi, S.T. Hedetniemi, and A.L. Liestman. A survey of gossiping and broadcasting in communication networks. *Networks*, 18:319–349, 1988.
- [11] J. Hromkovič, R. Klasing, B. Monien, and R. Peine. Dissemination of information in interconnection networks (broadcasting and gossiping). *Combinatorial Network Theory*, pages 125–212, 1995. eds. D.-Z. Zhu and D.F. Hsu (Kluwer Academic Publishers).
- [12] W. Knödel. New gossips and telephones. *Discrete Mathematics*, 13:95, 1975.
- [13] A.L. Liestman and J.G. Peters. Minimum broadcast digraphs. *Discrete Applied Mathematics*, 37/38:401–419, 1992.