Families of Graphs Having Broadcasting and Gossiping Properties

Guillaume Fertin, André Raspaud[1]

LaBRI U.M.R. C.N.R.S. 5800, Université Bordeaux I
351 Cours de la Libération, F33405 Talence Cedex
{fertin,raspaud}@labri.u-bordeaux.fr

Abstract. Broadcasting and gossiping are two problems of information dissemination described in a group of individuals connected by a communication network. In broadcasting (resp. gossiping), one node (resp. every node) has a piece of information and needs to transmit it to everyone else in the network. These communications patterns find their main applications in the field of interconnection networks for parallel architectures. In this paper, we are interested in Minimum Broadcast (resp. Gossip, Linear Gossip) Graphs (resp. Digraphs), that is graphs (resp. digraphs) that can achieve broadcasting (resp. gossiping, linear gossiping) in minimum time, and with a minimum number of edges. Many papers have investigated these subjects, but only a few general results on the size of graphs of order n are known. In this paper, we take the census of all the known non-isomorphic families of graphs (resp. digraphs) which are Minimum Broadcast Graphs, Minimum Gossip Graphs, Minimum Linear Gossip Graphs and/or Minimum Broadcast Digraphs, and we show that in most cases, the proposed minimum graphs that can be found in the literature are Knödel graphs [10, 7].

Keywords : Broadcasting, gossiping, minimum broadcast graphs, minimum gossip graphs, Knödel graphs, circulant graphs, hypercubes.

1 Introduction

Broadcasting and gossiping are two problems of information dissemination described in a group of individuals connected by a communication network. In *broadcasting* (resp. *gossiping*), one node (resp. every node) knows a piece of information and needs to transmit it to everyone else. This is achieved by placing communication calls over the communication lines of the network. Throughout this paper (except in one case, which, for readability reasons, is treated separately below), we will consider a *1-port* and *unit cost* model, that is a node can communicate with at most one of its neighbours at any given time, and a communication between two nodes takes one unit of time. This model implies that we will deal with connected graphs without loops or multiple edges

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to model the communication networks. Note also that, depending on the cases, we will either consider a *half-duplex* or a *full-duplex* model. In the latter, when a communication takes place along a communication line, the information flows in both directions, while in the former only one direction is allowed. Hence, in the *half-duplex* model, we will deal with directed graphs, while we will consider undirected graphs in the *full-duplex* model.

Let us first consider the *full-duplex* model. Let G be a graph modelling an interconnection network. We will denote by $b(v)$ the *broadcast time* of v, that is the time to achieve broadcasting from a vertex v of G in the network. Moreover, $b(G)$, the *broadcast time* of G, is defined as follows : $b(G) = max\{b(v) \mid v \in V(G)\}$. If we consider the complete graph of order n, K_n, it is not difficult to see that $b(K_n) = \lceil \log_2(n) \rceil$. Any graph G such that $b(G) = b(K_n) = \lceil \log_2(n) \rceil$ is called a *broadcast graph*. Note that it is not necessary to consider K_n to get a broadcast graph. Hence we call *Minimum Broadcast Graph* of order n, or MBG_n, any broadcast graph G having the minimum number of edges. This number is denoted by $B(n)$.

Similarly, in the *half-duplex* model, we have the following : $\boldsymbol{b}(v)$ is the *broadcast time of vertex* v and $\boldsymbol{b}(G) = max\{\boldsymbol{b}(v) \mid v \in V(G)\}$ is the *broadcast time of digraph* G. Liestman and Peters [13] have shown that $\boldsymbol{b}(K_n^*) = \lceil \log_2(n) \rceil$, where K_n^* is the complete directed graph of order n, that is the complete graph K_n where each undirected edge uv has been replaced by a pair of symmetric edges (u, v) and (v, u). Any digraph G such that $\boldsymbol{b}(G) = \boldsymbol{b}(K_n^*)$ is called a *broadcast digraph*, and, similarly to the undirected case, it is not necessary to consider K_n^* to get a broadcast digraph. Hence we will call *Minimum Broadcast Digraph* of order n, or MBD_n, any broadcast digraph with the minimum number of edges. This number is denoted by $\boldsymbol{B}(n)$.

The gossiping problem, back in the *full-duplex* model, relies on analogous definitions. Let $g(G)$ be the time to gossip in a graph G. Knödel [10] has shown that for the complete graph K_n, we have :

- $g(K_n) = \lceil \log_2(n) \rceil$ for any even n ;
- $g(K_n) = \lceil \log_2(n) \rceil + 1$ for any odd n ;

Any graph G such that $g(G) = g(K_n)$ is called a *gossip graph*. As previously, it is not necessary to consider K_n to get a gossip graph. Consequently, we call a *Minimum Gossip Graph* of order n, or MGG_n, any gossip graph with a minimum number of edges. This number is denoted by $G(n)$.

Remark 1.1. In this paper, we will deliberately not mention *Minimum Gossip Digraphs* (i.e. graphs achieving gossiping in the *half-duplex* model), since very little is known about these digraphs. In particular, no general result concerning its number of directed edges is known.

Now suppose we do not consider a *unit cost* model, but a *linear cost* one, that is each communication implies a fixed start-up time β, and a propagation time

τ proportional to the amount of information exchanged. We then define $g_{\beta,\tau}(G)$ to be the time to gossip in the graph G, and $g_{\beta,\tau}(n)$ to be the time to gossip in K_n. Fraigniaud and Peters [7] proved that $g_{\beta,\tau}(n) = \lceil \log_2(n) \rceil \beta + (n-1)\tau$ for any $\beta \geq 0$ and $\tau \geq 0$. A *Linear Gossip Graph* will denote any graph G able to gossip in $g_{\beta,\tau}(n)$, and a *Minimum Linear Gossip Graph*, or $MLGG$, is a Linear Gossip Graph with the minimum number of edges, noted $G_{\beta,\tau}(n)$.

In this paper, we intend to survey the general known results concerning the size - that is, the number of edges - of a MGG (resp. $MLGG$, MBG, MBD) with n nodes, and mainly to point out several non-isomorphic families of graphs which are MGG (resp. $MLGG$, MBG, MBD) of order n.

In Sect. 3, we will survey the general results concerning Minimum Broadcast Graphs of order n (e.g. $n = 2^k$ and $n = 2^k - 2$). We will first show three non-isomorphic families of graphs which are MBGs of order 2^k. Moreover, we will show that the examples of MBGs of order $2^k - 2$ given independently by Khachatrian et al. [9] and Dinneen et al. [2] are both isomorphic to the family of Knödel graphs of degree $k - 1$.

In Sect. 4, we will survey the known general results concerning Minimum Broadcast Digraphs of order n (e.g. $n = 2^k$, $n = 2^k - 1$ and $n = 2^k - 2$) and give different non-isomorphic families of digraphs which are MBDs of order n for these three cases. In particular, for $n = 2^k$, we give $k + 3$ non-isomorphic families of MBGs.

In Sect. 5, we will focus on Minimum Gossip Graphs. The size of such graphs is known in the general case for $n = 2^k$, $n = 2^k - 2$ and $n = 2^k - 4$ (see [11]). For each of the three cases, we will survey the results ; we will first show that, in the case $n = 2^k$, the three families of MBGs of order n given in Sect. 3 remain MGGs. Moreover, we will show that the family of graphs pointed out by Labahn [11] as MGGs of order $2^k - 2$ (resp. $2^k - 4$) is in fact the family of Knödel graphs of degree $k - 1$.

Section 6 will be devoted to $MLGGs$. We will survey the results of [7], and we will show that, for $n = 2^k$, the three non-isomorphic families which are MGGs remain $MLGG$s. Moreover, for even n such that $2^k - 6 \leq n \leq 2^k - 2$ with $k \geq 4$, the Knödel graphs of degree $k - 1$ are $MLGG$s.

Finally, Sect. 7 recalls the different results given in the paper and gives a general method to obtain recursively families of MBGs (resp. MBDs, MGGs, $MLGG$s) of order 2^k, families of MBDs of order $2^k - 2$ and families of MGGs (resp. $MLGG$s) of order $2^k - 4$.

2 Definitions

In the following, we will consider different families of graphs that have been defined in the literature, and we will show they are either MGGs, $MLGG$s, MBGs and/or MBDs. In this section, we give the definitions of each of the family we are going to use.

Definition 2.1 (Hypercube of dimension k). *The hypercube of dimension k, H_k, has $n = 2^k$ vertices. Each vertex of the vertex set V is of the form $(x_0, x_1, \ldots, x_{k-1})$ with $x_i \in \{0, 1\}$ for any $0 \leq i \leq k-1$. The edge set E is of cardinality $k \cdot 2^{k-1}$ and is of the form $E = \{((x_0, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{k-1}), (x_0, x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{k-1}))\}$ for every $i \in \{0, \ldots, k-1\}$.*

Definition 2.2 (Circulant graphs and digraphs).

- *A* circulant graph *on n vertices $C_n(a_1, a_2, \ldots a_p)$ ($a_i \in \mathbb{N}^*$), $a_1 < a_2 < \ldots < a_p$, has vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E = \{(v_x, v_y) \mid \exists\ a_i, 1 \leq i \leq p$ such that $x + a_i \equiv y \ (mod\ n)\}$.*
- *A* circulant digraph *on n vertices $C'_n(a_1, a_2, \ldots, a_p)$ ($a_i \in \mathbb{N}^*$), $a_1 < a_2 < \ldots < a_p$, has vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E = \{(v_x, v_y) \mid \exists\ a_i, 1 \leq i \leq p$ such that $x + a_i \equiv y \ (mod\ n)\}$.*

Remark 2.1. In this paper, we will often focus on circulant digraphs, and particularly on some specific families of such digraphs, for instance $C'_{2^k}(1, 3, \ldots, 2^k - 1)$ [16, 5]. We refer to Fig. 1 for some examples of circulant digraphs. Note also that we will be interested by a certain family of circulant (undirected) graphs, namely $G(n, d)$, which is defined below.

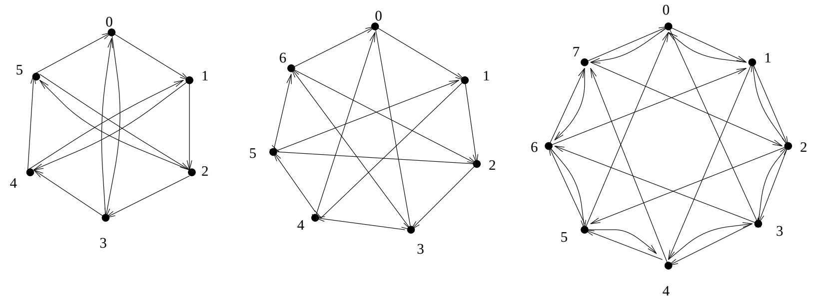

Fig. 1. $C'_6(1, 3)$, $C'_7(1, 3)$ and $C'_8(1, 3, 7)$ (from left to right)

Definition 2.3 (Circulant Graphs $G(n, d)$ [17]). *The circulant graph $G(n, d)$ with $d \geq 2$, is defined as follows. The vertex set is $V = \{0, 1, 2, \ldots n-1\}$, and the edge set is $E = \{(u, v) \mid \exists i, 0 \leq i \leq \lceil log_d(n) \rceil - 1$, such that $u + d^i \equiv v \ (mod\ n)\}$.*

Remark 2.2. Note that $G(n, d)$ is a circulant graph $C_n(d^0, d^1, \ldots d^{\lceil log_d(n) \rceil - 1})$. In [17], Park and Chwa proved that if $n = cd^m$ with $1 \leq c < d$, then $G(n, d)$ can be constructed recursively, using d copies of $G(cd^{m-1}, d)$. Note also that for our purpose, we will consider the family of circulant graphs $G(2^k, 4)$, such as the one displayed in Fig. 2. In that case, $G(2^k, 4)$ can be considered as a $G(cd^m, d)$, where $c = 1$ when k is even ($k = 2m$), and $c = 2$ when k is odd ($k = 2m + 1$).

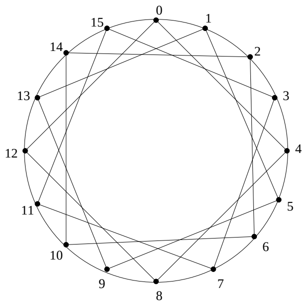

Fig. 2. The recursive circulant graph $G(16,4)$

Definition 2.4 (Knödel graph $W_{\Delta,n}$ [10, 7]). *The* Knödel graph *on $n \geq 2$ vertices (n even) and of maximum degree $\Delta \geq 1$ is denoted $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the couples (i,j) with i=1,2 and $0 \leq j \leq \frac{n}{2}-1$. For every j, $0 \leq j \leq \frac{n}{2}-1$, there is an edge between vertex $(1,j)$ and every vertex $(2, j+2^k-1$ mod $\frac{n}{2})$, for $k = 0, \ldots, \Delta-1$.*

Remark 2.3. For $0 \leq k \leq \Delta - 1$, an edge of $W_{\Delta,n}$ which connects a vertex $(1,j)$ to the vertex $(2, j+2^k-1 \bmod \frac{n}{2})$ is said to be *in dimension k*.

Note that when $\Delta \geq 2$, the Knödel graphs can also be defined as Cayley graphs on the semi-direct product $G = \mathbb{Z}_{\frac{n}{2}} \ltimes \mathbb{Z}_2$ for the multiplicative law : $(x,y)(x',y') = (x+x', y+(-1)^x y')$, with $x, x' \in \mathbb{Z}_2$ and $y, y' \in \mathbb{Z}_{\frac{n}{2}}$, and with the set of generators $S = \{(1, 2^i - 1), 0 \leq i \leq \Delta - 1\}$ [8].

In the following, we will be mostly interested by $W_{k-1,2^k-4}$, $W_{k-1,2^k-2}$ and $W_{k,2^k}$. For a better understanding, we give two examples of such Knödel graphs (cf. Fig. 3).

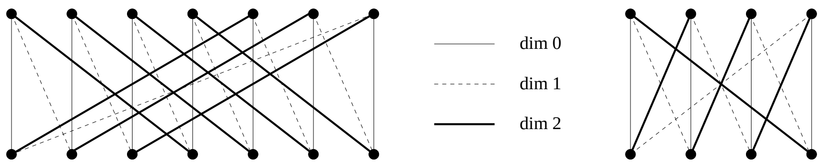

Fig. 3. $W_{3,14}$ and (right) $W_{3,8}$

3 Minimum Broadcast Graphs

3.1 General Results

Let us first recall the only known general results that exist concerning MBGs.

Theorem 3.1. *Let $B(n)$ be the size of a MBG of order n. Then :*

- $B(2^k) = k \cdot 2^{k-1}$ *for all k [4];*
- $B(2^k - 2) = (k-1) \cdot (2^{k-1} - 1)$ *for all $k \geq 3$ [2, 9].*

3.2 Non-Isomorphic Families of MBGs of Order $n = 2^k$

Thanks to the results above, we can prove the following Theorem.

Theorem 3.2. *There are at least three non-isomorphic families of graphs which are $MBGs$ of order $n = 2^k$ for any $k \geq 4$. They are the following :*

- *The hypercube of dimension k, H_k ;*
- *The recursive circulant graph $G(2^k, 4)$;*
- *The Knödel graph $W_{k,n}$.*

Proof. We need to prove first that each family of graphs is MBG of order 2^k. As seen in the previous Theorem, the size $B(n)$ of a MBG of order $n = 2^k$ verifies $B(2^k) = k \cdot 2^{k-1}$ [4]. Since the broadcast time and the gossip time are equal when the number of nodes is even, and since $B(2^k) = G(2^k)$ for any k [11], we conclude that all the graphs which are MGG_{2^k} are MBG_{2^k} as well. Since it is well-known that those three families of graphs are MGGs of order 2^k (cf. for this Proof of Theorem 5.2), we know that they are also MBGs.

Now let us prove that these three families of graphs are not isomorphic. It is easy to see that $G(2^k, 4)$ is not isomorphic to H_k for any $k \geq 2$ since, in that case, their diameter differ. Indeed, Park and Chwa [17] have proved that $Diam(G(2^k, 4)) = \lceil \frac{3k-1}{4} \rceil$, while it is well-known that $Diam(H_k) = k$.

Similarly, it is not difficult to see that $W_{k,2^k}$ and $G(2^k, 4)$ are not isomorphic for any $k \geq 3$. Indeed, Knödel graphs are bipartite by definition, and it is well known that there cannot exist cycles of odd length in bipartite graphs (see for instance [1], p.6). However, there always exists cycles of odd length in $G(2^k, 4)$ when $k \geq 3$. For instance, consider the cycle of length 5, $(0-1-2-3-4-0)$.

Finally, we prove that $W_{k,2^k}$ and H_k are not isomorphic for any $k \geq 4$. Indeed, it is known that hypercubes are $(0,2)$-graphs, that is any two vertices which are at distance 2 have exactly two neighbours in common [15, 12]. Now let us consider the vertices $u = (1,1)$ and $v = (1, 2^{k-1} - 2)$ in $W_{k,2^k}$. Those two vertices cannot be neighbours by definition, since the graph is bipartite. Actually, they are at distance 2, because there is an edge $(1,1)(2,1)$ (dimension 0) and an edge $(2,1)(1, 2^{k-1} - 2)$ (dimension 2). Let N_u (resp. N_v) be the set of neighbours of u (resp. of v) in $W_{k,2^k}$. We have :

- $N_u = (2,0) \cup \{(2, 2^p) \mid 0 \leq p \leq k-2\}$;
- $N_v = (2, 2^{k-1} - 2) \cup (2, 2^{k-1} - 1) \cup \{(2, 2^p - 3) \mid 2 \leq p \leq k-1\}$.

In that case, let us count the number of neighbours u and v have in common. Standard calculations show that, for any $k \geq 4$, u and v have only one neighbour in common, which is $(2,1)$. Hence $W_{k,2^k}$ is not a $(0,2)$-graph, which yields that it is not isomorphic to H_k for any $k \geq 4$. □

Remark 3.1. In their paper, Fraigniaud and Peters [7] said that $W_{k,2^k}$ and H_k were not isomorphic for any $k \geq 4$, saying that there are no 4-cycles in $W_{k,2^k}$ with $k \geq 4$. Though the result is correct, the proof is not since there are always 4-cycles in $W_{k,2^k}$ for any $k \geq 2$. Take for instance the cycle $(1,0)-(2,1)-(1,1)-(2,0)-(1,0)$.

3.3 Families of MBGs of Order $n = 2^k - 2$

Khachatrian et al. [9] and Dinneen et al. [2] have shown independently that the size of a MBG of order $n = 2^k-2$ for any $k \geq 3$ is $B(2^k-2) = (k-1)\cdot(2^{k-1}-1)$. For this they used different proofs, but we can show that the family of graphs they gave as MBGs is in both cases isomorphic to the Knödel graph $W_{k-1,2^k-2}$. Hence the following propositions.

Proposition 3.1. *The family of graphs given in [9] as $MBGs$ on 2^k-2 vertices is isomorphic to the Knödel graphs $W_{k-1,2^k-2}$.*

Proof. In [9], the authors gave a family of graphs which were MBGs of order $n = 2^k - 2$. Let KH_n be this family of graphs. It is defined as follows. Let the vertex set be $V = \{v_0, v_1, \ldots v_{2^k-3}\}$, and the edge set $E = \{(v_i, v_j) \mid i+j \equiv 2^r-1 \mod (n)\}$ with $r \in \{1, 2, \ldots k-1\}$. Let us prove that this definition is equivalent to the one of $W_{k-1,2^k-2}$. First, note that KH_n is a bipartite graph : indeed, we can partition the set of vertices V in V_1 and V_2 as follows : let $V_1 = \{v_{2p} \mid 0 \leq p \leq \frac{n}{2}-1\}$ and $V_2 = \{v_{2p+1} \mid 0 \leq p \leq \frac{n}{2}-1\}$. Let us now identify V_i of KH_n ($i \in \{1,2\}$) to the sets of vertices $V_i' = \{(i,p) \mid 0 \leq p \leq \frac{n}{2}-1\}$ of $W_{k-1,n}$. For this, let us rename the vertices v_i ($i \in \{0, \ldots n-1\}$) of V in KH_n the following way :

- Let us rename v_{2p} as the couple $(1, \frac{n}{2} - p \mod \frac{n}{2})$ for every $0 \leq p \leq \frac{n}{2} - 1$;
- Let us rename v_{2p+1} as the couple $(2, p)$ for every $0 \leq p \leq \frac{n}{2} - 1$;

In that case, it is not difficult to see that KH_n and $W_{k-1,n}$ are isomorphic. Indeed, let us come back to the definition of Khachatrian et al., i.e. there is an edge (v_i, v_j) iff $i + j \equiv 2^r - 1 \mod n$, with $r \in \{1, 2, \ldots k-1\}$. W.l.o.g., let us consider i even, and consequently j odd. Let then $i = 2m$ and $j = 2l+1$. By definition, v_i and v_j are neighbours iff $2m + 2l + 1 \equiv 2^r - 1 \mod n$ for some $r \in \{1, \ldots k-1\}$. That is $m+l \equiv 2^{r'} - 1 \mod \frac{n}{2}$ for some $r' \in \{0, \ldots, k-2\}$. Now let us replace v_i and v_j by their identifications above. We conclude that there is an edge $((1,m),(2,l))$ iff $l \equiv m + 2^{r'} - 1 \mod \frac{n}{2}$ for some $r' \in \{0, \ldots, k-2\}$. This is exactly the definition of the Knödel graph $W_{k-1,n}$. □

Proposition 3.2. *The family of graphs given in [2] as $MBGs$ on 2^k-2 vertices is isomorphic to the Knödel graphs $W_{k-1,2^k-2}$.*

Proof. Dinneen et al. [2] gave MBGs of order $2^k - 2$ by defining them as Cayley graphs on the dihedral group $D_{\frac{n}{2}}$ (with $n = 2^k - 2$), with generators $\alpha, \alpha\beta^1, \alpha\beta^3, \ldots, \alpha\beta^{2^{k-2}-1}$, and where $\alpha^2 = e$, $\alpha\beta\alpha^{-1} = \beta^{-1}$ and $\beta^{2^{k-1}-1} = e$.

To prove that this Cayley graph is isomorphic to $W_{k-1,2^k-2}$, it suffices to prove that this is a semi-direct product $\mathbb{Z}_{\frac{n}{2}} \ltimes \mathbb{Z}_2$ (with $n = 2^k - 2$) as defined in Remark 2.3. Identifiying $\alpha\beta^i$ to $(1, i)$ and β^i to $(0, \frac{n}{2} - i \bmod (\frac{n}{2}))$ for all $0 \leq i \leq \frac{n}{2} - 1$ gives us directly the result. □

4 Minimum Broadcast Digraphs

4.1 General Results

Let us first recall the only known general results that exist concerning MBDs.

Theorem 4.1. *Let $\boldsymbol{B}(n)$ be the size of a MBD of order n. Then :*

- $\boldsymbol{B}(2^k) = k \cdot 2^k$ *for all k [13];*
- $\boldsymbol{B}(2^k - 1) = (k-1) \cdot (2^k - 1)$ *for all $k \geq 3$ [5];*
- $\boldsymbol{B}(2^k - 2) = (k-1) \cdot (2^k - 2)$ *for all $k \geq 3$ [5].*

4.2 Non-Isomorphic Families of MBDs of Order n for $2^k - 2 \leq n \leq 2^k$

Since we deal with directed graphs in this section, we need to introduce the following notion.

Definition 4.1. *Let us consider an undirected graph G. We will denote by G^* the directed graph obtained by replacing each edge of G by two arcs in opposite directions.*

Before proving the main Theorem (Theorem 4.3), let us focus on a special case of circulant digraphs of order $n = 2^k$, and let us show that they are MBDs. This is done in Theorem 4.2.

Theorem 4.2. *Let $n = 2^k$ and let $a_i = 2^i$ for all $0 \leq i \leq k-1$. Let us choose $k-1$ distinct a_i among the k existing ones. We call them the b_j $(1 \leq j \leq k-1)$, such that $b_1 < b_2 \ldots < b_{k-1}$. Then $C'_n(1, 1+b_1, 1+b_1+b_2, \ldots, 1+\sum_{j=1}^{k-1} b_j)$ is a MBD.*

Proof. First, note that each of those digraphs are of size $k \cdot 2^k$. Since $\boldsymbol{B}(2^k) = k \cdot 2^k$ by Theorem 4.1, it remains to show that any vertex can broadcast in minimum time in each of these digraphs. This result can be proved by induction on k. When $k = 2$, we have $a_0 = 1$ and $a_1 = 2$. Then we can build two circulant digraphs, namely $C'_4(1, 2)$ (when $b_1 = a_0$) and $C'_4(1, 3)$ (when $b_1 = a_1$). It is not difficult to see that those two digraphs are MBDs. Now suppose that the Theorem is true for k, and let us show it is then true for $k + 1$. The key idea here is to partition the vertex set in two distinct subsets, such that each of the digraphs induced by each of the subsets is one of the MBDs of order 2^k constructed as above. For this, let us distinguish two cases :

- $b_1 = a_1 = 2$. In that case, we know that $b_i = a_i$ for any $1 \leq i \leq k$, that is the circulant digraph is $C'_{2^{k+1}}(1, 3, 7 \ldots 2^{k+1} - 1)$. In [5], it has been shown that this family of circulant digraphs is MBD for any k.
- $b_1 = a_0 = 1$. In that case, we can see that the digraph constructed will be of the form : $C'_{2^{k+1}}(1, 2, \alpha_1, \ldots, \alpha_{k-1})$, where each of the α_i is even. Let us then partition the set V of vertices in two distinct subsets V_0 and V_1 as follows : $V_0 = \{v_{2i} \mid 0 \leq i \leq 2^{k-1} - 1\}$ and $V_1 = \{v_{2i+1} \mid 0 \leq i \leq 2^{k-1} - 1\}$. It is interesting to note that, in that case, the digraph induced by V_0 (resp. V_1) is isomorphic to one of the circulant digraphs of order 2^k constructed by our method. Indeed, let us distinguish two more cases.
  1. If $b_2 = a_2 = 4$, the circulant digraph constructed will be $C'_{2^{k+1}}(1, 2, 6, 14, \ldots, 2^{k+1} - 2)$, and the digraph induced by V_0 (resp. V_1) is $C'_{2^k}(1, 3, 7, \ldots 2^k - 1)$.
  2. Otherwise, the digraph induced by V_0 (resp. V_1) is the circulant digraph of order 2^k constructed with the parameters $b'_i = \frac{b_{i+1}}{2}$ for all $1 \leq i \leq k$.

In all cases, we see that either the circulant digraph constructed is $C'_{2^{k+1}}(1, 3, 7 \ldots 2^{k+1} - 1)$, which is known to be MBD, or that it can be recursively constructed with two copies of a circulant digraph of order 2^k (constructed by our method) joined by the following Hamiltonian circuit : $v_0 - v_1 - v_2 - \ldots - v_{2^{k+1}-1} - v_0$. Since we supposed that those two copies are MBDs of order 2^k, we can prove now the Theorem : indeed, we deal with circulant digraphs, hence they are vertex-transitive. Consequently, we can focus on broadcasting from one particular vertex, say v_0. In that case, the broadcast scheme from v_0 is the following : first, broadcast to v_1 during the first time unit. This is possible since there is a directed edge (v_0, v_1) by definition. Now, since $v_0 \in V_0$ and $v_1 \in V_1$, let v_0 (resp. v_1) broadcast in the copy of the MBD of order 2^k induced by V_0 (resp. V_1) between time units 2 and $k+1$. This completes broadcasting from v_0 in $k+1$ time units. Hence, each of those digraphs of dimension $k + 1$ is a MBD. □

Proposition 4.1. *For $n = 2^k$, there exist k circulant digraphs constructed as in Theorem 4.2. Moreover, those k digraphs are non-isomorphic one to another.*

Proof. Since we choose $k - 1$ distinct b_i among k a_i, it directly follows that we have k possibilities, hence we can construct k circulant digraphs of order 2^k. Let us call them the $C'^{i}_{2^k}$, with $1 \leq i \leq k$, such that $a_{i-1} \notin \{b_j \mid 1 \leq j \leq k - 1\}$. To prove that any two such digraphs are non-isomorphic, let us focus on those directed edges (u, v) such that there is a directed edge (v, u). Let us call them *symmetric edges*. Let us now consider a circulant digraph $C'^{i}_{2^k}$; let us delete all the non-symmetric directed edges, and replace all the pairs of symmetric edges by a single undirected edge. In that case, if we show that the (undirected) graph $G^i_{2^k}$ obtained this way is non-isomorphic to any of the other $G^j_{2^k}$ for $j \neq i$, it follows directly that each $C'^{i}_{2^k}$ is non-isomorphic to any $C'^{j}_{2^k}$ with $j \neq i$.

Let us show, by induction on k, that $G^{i+1}_{2^k}$ is composed of 2^i copies of cycles of length 2^{k-i} for all $0 \leq i \leq k-1$ (note that , when $i = k-1$, the cycle of length 2 will be considered here to be K_2). When $k = 2$, C'^{1}_4 is $C'_4(1, 3)$ and C'^{2}_4 is $C'_4(1, 2)$.

Then G_4^1 is the cycle of length 4, and G_4^2 is composed of two copies of the complete graph K_2. Now suppose the hypothesis is true for any k, and let us prove it for $k+1$. As seen in the proof of Theorem 4.2, k of the $k+1$ circulant digraphs of order 2^{k+1} can be built using two copies of a circulant digraph of order 2^k, and joining them by the hamiltonian circuit $v_0 - v_1 - v_2 - \ldots - v_{2^{k+1}-1} - v_0$; the $k+1$-th digraph being $C'_{2^{k+1}}(1, 3, 7, \ldots 2^{k+1}-1)$. For each $1 \leq i \leq k$, we can see that $C'^{\,i+1}_{2^{k+1}}$ is the digraph constructed using two copies of $C'^{\,i}_{2^k}$, and that $C'^{\,1}_{2^{k+1}}$ is $C'_{2^{k+1}}(1, 3, 7, \ldots 2^{k+1}-1)$. In that case, we know by construction that $G^1_{2^{k+1}}$ is the cycle of length 2^{k+1}, and since each $C'^{\,i}_{2^{k+1}}$ is constructed from two copies of $C'^{\,i-1}_{2^k}$ $(2 \leq i \leq k+1)$ joined by a hamiltonian circuit, $G^i_{2^{k+1}}$ will be the union of two copies of $G^{i-1}_{2^k}$ for all $2 \leq i \leq k+1$. Hence each $G^i_{2^{k+1}}$ is composed of 2^{i-1} copies of cycles of length $2^{k+1-(i-1)}$ for all $1 \leq i \leq k+1$, and the result is proved by induction. □

Thanks to the Theorem and Proposition above, and to some results given in [13, 17, 5], we have the following Theorem.

Theorem 4.3 ([13, 17, 5]). *For the following values of n, the families of digraphs listed below are non-isomorphic families of $MBDs$:*

- *For $n = 2^k$, the directed hypercube H_k^*, the directed Knödel graph $W_{k,n}^*$, the directed recursive circulant graph $G^*(n, 4)$, and the k circulant digraphs $C'^{\,i}_n$ with $1 \leq i \leq k$ are $MBDs$;*
- *For $n = 2^k - 1$, the circulant digraph $C'_n(1, 3, \ldots 2^{k-1}-1)$ is a MBD ;*
- *For $n = 2^k - 2$, the directed Knödel graph $W_{k-1,n}^*$ and the circulant digraph $C'_n(1, 3, \ldots 2^{k-1}-1)$ are $MBDs$.*

Proof. Most of these properties have already been shown in [13], [5], Theorem 4.2 and Proposition 4.1. Some others remain to be proved, as done in the following. Let us prove first that $G^*(2^k, 4)$ is a MBD. This is straightforward since $\boldsymbol{B}(2^k) = k \cdot 2^k$, and since $\boldsymbol{b}(G^*(2^k, 4)) = b(G(2^k, 4))$ for any n (indeed, the broadcast scheme used in $G(2^k, 4)$ remains valid in $G^*(2^k, 4)$).

Now let us prove that any $C'^{\,i}_{2^k}$ $(1 \leq i \leq k)$ is not isomorphic to H_k^* (resp. $G^*(2^k, 4)$, $W^*_{k,2^k}$). For this, note that for any directed edge (u, v) in H_k^* (resp. $G^*(2^k, 4)$, $W^*_{k,2^k}$), there exists a directed edge (v, u) by definition. However, in $C'^{\,1}_{2^k}$, there is a directed edge (v_0, v_3) and no directed edge (v_3, v_0) for any $k \geq 3$; moreover, in any $C'^{\,i}_{2^k}$ with $2 \leq i \leq k$, there is a directed edge (v_0, v_1) and no directed edge (v_1, v_0) for any $k \geq 2$. Hence there is no isomorphism between H_k^* (resp. $G^*(2^k, 4)$, $W^*_{k,2^k}$) and $C'^{\,i}_{2^k}$ for any $1 \leq i \leq k$. This completes the proof of the Theorem. □

5 Minimum Gossip Graphs

5.1 General Results

Let us first recall the only known general results that exist concerning MGGs. These results can be found in [11].

Theorem 5.1. *Let $G(n)$ be the size of a MGG of order n. Then :*

- $G(2^k) = k \cdot 2^{k-1}$ *for all* k ;
- $G(2^k - 2) = (k-1) \cdot (2^{k-1} - 1)$ *for all* $k \geq 4$;
- $G(2^k - 4) = (k-1) \cdot (2^{k-1} - 2)$ *for all* $k \geq 6$;

5.2 Non-Isomorphic Families of MGGs on 2^k vertices

Thanks to the results above, and thanks to Theorem 3.2, we can prove the following Theorem.

Theorem 5.2. *There are at least three non-isomorphic families of graphs which are MGGs of order $n = 2^k$ for any $k \geq 4$. They are the following :*

- *The hypercube of dimension k, H_k ;*
- *The recursive circulant graph $G(2^k, 4)$;*
- *The Knödel graph $W_{k,n}$.*

Proof. In Theorem 3.2, the non-isomorphism of those three families has been proved. It remains to prove here that those three families of graphs are MGGs of order 2^k. For the hypercube, it is a very well-known property (see for instance [11]). The recursive circulant $G(2^k, 4)$ is also a MGG. For this, we refer to [17, 14]. Moreover, $W_{k,2^k}$ is also a MGG on $n = 2^k$ nodes, since $W_{k,2^k}$ is underlying Knödel's proof [10] that it is possible to gossip in $\lceil \log_2(n) \rceil$ time units when n is even. □

5.3 Families of MGGs of Order $n = 2^k - 2$ and $n = 2^k - 4$

In [11], Labahn proved the exact value of $G(n)$ for $n = 2^k - 2$ and $n = 2^k - 4$. For this, he displayed graphs which were gossip graphs and had the minimum number of edges. It appears that these graphs he gave as examples of MGGs are isomorphic to the Knödel graphs $W_{k-1,n}$, as proved in the following Proposition.

Proposition 5.1.
The family of graphs given in [11] as MGGs on n vertices :

- *Is isomorphic to the Knödel graphs $W_{k-1,2^k-2}$ when $n = 2^k - 2$;*
- *Is isomorphic to the Knödel graphs $W_{k-1,2^k-4}$ when $n = 2^k - 4$.*

Proof. In [11], Labahn gave a family of graphs that were MGGs of order $n = 2^k - 2$. He pointed out that these graphs are isomorphic to the Cayley graphs on the dihedral group $D_{\frac{n}{2}}$ that Dinneen et al. [2] gave as examples of MBGs of the same order. By Proposition 3.2, the result follows directly : this graph is isomorphic to the Knödel graph $W_{k-1,2^k-2}$.

In the case $n = 2^k - 4$, Labahn [11] showed that the graphs he gave as MGGs of order n are Cayley graphs defined on the dihedral group $D_{\frac{n}{2}}$, this time with $n = 2^k - 4$, and with the same set of generators as above. Since this remains a semi-direct product as defined in Remark 2.3 (the only difference here being that $n = 2^k - 4$), it follows that this graph is isomorphic to the Knödel graph $W_{k-1,2^k-4}$. □

6 Minimum Linear Gossip Graphs ($MLGGS$)

These graphs have been studied by Fraigniaud and Peters [7]. They proved that $G_{\beta,\tau}(n) = G_{1,1}(n)$ for any $\beta > 0$ and $\tau > 0$, that is the structure of a $MLGG$ does not depend on β and τ. In this section, we consider $\beta \neq 0$ and $\tau \neq 0$. Fraigniaud and Peters proved the following.

Theorem 6.1 ([7]). *For the following values of even n, we have :*

- *For $n = 2^k$, $G_{\beta,\tau}(n) = k \cdot 2^{k-1}$, and H_k and $W_{k,2^k}$ are $MLGG$s ;*
- *For all $2^k - 6 \leq n \leq 2^k - 2$ with $k \geq 4$, $G_{\beta,\tau}(n) = \frac{n(k-1)}{2}$ and $W_{k-1,n}$ is a $MLGG$.*

Moreover, we have the following proposition.

Proposition 6.1. *For all $n = 2^k$, $G(2^k, 4)$ is a Minimum Linear Gossip Graph.*

Proof. It appears that the scheme used in [14] to achieve gossiping in minimum time in the *unit cost* model in $G(2^k, 4)$ also allows to achieve gossiping in minimum time in the *linear cost* model. This can be proved by induction on n. Indeed, it is not difficult to see that it is possible to gossip in minimum time, that is in $g_{\beta,\tau}(2^k) = k\beta + (2^k - 1)\tau$, in $G(2^k, 4)$ when $k = 1$ and $k = 2$.

Now suppose it is possible to gossip in time $g_{\beta,\tau}(2^k)$ in $G(2^k, 4)$ for some $k \geq 1$. We use the fact that $G(2^{k+2}, 4)$ can be built from four copies of $G(2^k, 4)$, as stated in [17, 14]. Let us gossip independently in each of the four copies of $G(2^k, 4)$. This takes $g_{\beta,\tau}(2^k)$, and at the end of the process, each vertex of each copy G_i ($i \in [1; 4]$) of $G(2^k, 4)$ knows all the information contained in its own copy. During the next step, let each vertex of G_0 (resp. of G_2) communicate with the vertex of G_1 (resp. of G_3) to which it is adjacent. This takes $\beta + 2^k\tau$, and, after that round, each vertex knows 2^{k+1} different pieces of information. Now, during the last step, let each vertex of G_0 (resp. of G_1) communicate with the vertex of G_2 (resp. of G_3) to which it is adjacent. This takes $\beta + 2^{k+1}\tau$, and at the end of this step, gossiping is completed.

On the whole, the time used to gossip in $G(2^{k+2}, 4)$ is $g_{\beta,\tau}(2^k) + \beta + 2^k\tau + \beta + 2^{k+1}\tau$. Since $g_{\beta,\tau}(2^k) = k\beta + 2^k\tau$ by hypothesis, we know that gossiping can be achieved in minimum time in $G(2^{k+2}, 4)$. This proves the result by induction. □

7 Summary of the Results

The different results concerning non-isomorphic families of graphs (resp. digraphs) which are MBG, MBDs, MGGs and/or $MLGG$s are displayed in Figs. 4 and 5. In addition to these results, we can note the following : in the undirected case, suppose we have two graphs of order n which are MBGs, namely G_1 and G_2. If $B(2n) = 2 \cdot B(n) + n$, then any graph obtained by joining the vertices of G_1 to the vertices of G_2 by a perfect matching is still a MBD of order $2n$. Indeed, it has $B(2n)$ edges, and any vertex will be able to broadcast in minimum time (broadcast through the edge of the perfect matching first, then broadcast in each of the two MBGs of order n). This is similar to Farley's construction [3], except that the perfect matching here could correspond to any permutation of the indices between the n vertices u_i of G_1 and the n vertices v_i of G_2 $(1 \leq i \leq n)$. In particular, when $n = 2^k$, this method can be applied. Moreover, it can be applied recursively, with the possibility to take different permutations from one step to another. This allows us to get many non-isomorphic MBGs of order 2^k. It is easy to see that this method also works for $n = 2^k$ in the case of MGGs and $MLGG$s (in that case, it is necessary to gossip first in each copy, before using the last time unit to exchange information using the edges of the perfect matching).

This method also works in the directed case for MBDs, still with $n = 2^k$. In that case, instead of considering a perfect matching between two copies, it is necessary to consider two perfect matchings : one with arcs going from G_1 to G_2, the other with arcs going from G_2 to G_1.

Finally, we also see that this method can be applied, this time only once, when $n = 2^k - 2$ to obtain MGGs and $MLGG$s of order $2^{k+1} - 4$, and when $n = 2^k - 1$ to obtain MBDs of order $2^{k+1} - 2$.

Hence we have the following Proposition.

Proposition 7.1.

- *In the case $n = 2^k$, the method described above gives us recursively families of $MBGs$ (resp. $MBDs$, $MGGs$, $MLGGs$) of order 2^{k+1} ;*
- *In the case $n = 2^k - 1$, the method above, used once, gives us families of $MBDs$ of order $2^{k+1} - 2$;*
- *In the case $n = 2^k - 2$, the method described above, used once, gives us families of $MGGs$ (resp. $MLGGs$) of order $2^{k+1} - 4$.*

Note that the problem of determining how many of the families constructed as above are non-isomorphic is still open. However, since there exists $n!$ permutations on n vertices, and since it is possible to use different permutations when the graphs are constructed recursively, one may think that the number of non-isomorphic graphs obtained is relatively high.

8 Conclusion

This paper aims to gather information concerning the known general values of $G(n)$, $G_{\beta,\tau}(n)$, $B(n)$ and $\boldsymbol{B}(n)$, and above all, to give in each possible case as many non-isomorphic families of graphs which are MGGs, $MLGG$s, MBGs and/or MBDs as possible. This has been done by gathering the results from various authors. Moreover, it appears that, in the undirected case, for $n = 2^k - 2$ and $n = 2^k - 4$, the families of graphs given by the authors (namely [11] for the MGGs, and [9] and [2] for the MBDs) are always isomorphic to $W_{k-1,n}$. Though we know very little about the size of MGGs, $MLGG$s, MBGs and MBDs in

	Gossiping			
	MGG		$MLGG$	
n	$G(n)$	Graphs	$G_{\beta,\tau}(n)$	Graphs
2^k	$\frac{nk}{2}$	H_k $G(2^k,4)$ $W_{k,2^k}$	$\frac{nk}{2}$	H_k $G(2^k,4)$ $W_{k,2^k}$
2^k-2	$\frac{n(k-1)}{2}$	$W_{k-1,2^k-2}$	$\frac{n(k-1)}{2}$	$W_{k-1,2^k-2}$
2^k-4	$\frac{n(k-1)}{2}$	$W_{k-1,2^k-4}$	$\frac{n(k-1)}{2}$	$W_{k-1,2^k-4}$
2^k-6			$\frac{n(k-1)}{2}$	$W_{k-1,2^k-6}$

Fig. 4. Sum-up of the results : Gossiping

	Broadcasting			
	MBG		MBD	
n	$B(n)$	Graphs	$\boldsymbol{B}(n)$	Graphs
2^k	$\frac{nk}{2}$	H_k $G(2^k,4)$ $W_{k,2^k}$	nk	H_k^* $G^*(2^k,4)$ $W_{k,2^k}^*$ $C'^{\,i}_{2^k}$ with $1 \leq i \leq k$
2^k-1			$n(k-1)$	$C'_{2^k-1}(1,3,\ldots 2^{k-1}-1)$
2^k-2	$\frac{n(k-1)}{2}$	$W_{k-1,2^k-2}$	$n(k-1)$	$W_{k-1,2^k-2}^*$ $C'_{2^k-2}(1,3,\ldots 2^{k-1}-1)$

Fig. 5. Sum-up of the results : Broadcasting

the general case, we found it interesting to sum-up the results and give, as far as we know, all the non-isomorphic families of graphs that respect these properties.

It is interesting to note too that Knödel graphs seem to play an important role in these communications patterns, since they are MGGs, $MLGG$s, MBGs and MBDs in every known case of even order. Moreover, we believe that this family of graphs has many more interesting characteristics, such as the fact that $W_{k-2,n}$ is a gossip (hence broadcast) graph for any even n such that $2^{k-1}+2 \leq n \leq 3 \cdot 2^{k-2}-4$ (cf. [6]).

In a word, we believe our study could be useful as a handbook of non-isomorphic graphs being either MGGs, $MLGG$s, MBDs and/or MBDs.

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