On the S-Labeling problem

Guillaume Fertin

Laboratoire d'Informatique de Nantes-Atlantique (LINA), UMR CNRS 6241
Université de Nantes, 2 rue de la Houssinière, 44322 Nantes Cedex 3 - France
guillaume.fertin@univ-nantes.fr

Stéphane Vialette

IGM-LabInfo, CNRS UMR 8049, Université Paris-Est,
5 Bd Descartes 77454 Marne-la-Vallée, France
vialette@univ-mlv.fr

Abstract

Let G be a graph of order n and size m. A *labeling* of G is a bijective mapping $\theta : \mathbf{V}(G) \rightarrow \{1, 2, \ldots, n\}$, and we call $\Theta(G)$ the set of all labelings of G. For any graph G and any labeling $\theta \in \Theta(G)$, let $\mathsf{SL}(G, \theta) = \sum_{e \in \mathbf{E}(G)} \min\{\theta(u) : u \in e\}$. In this paper, we consider the S-Labeling problem, defined as follows: Given a graph G, find a labeling $\theta \in \Theta(G)$ that minimizes $\mathsf{SL}(G, \theta)$. The S-Labeling problem has been shown to be **NP**-complete [Via06]. We prove here basic properties of any optimal S-labeling of a graph G, and relate it to the Vertex Cover problem. Then, we derive bounds for $\mathsf{SL}(G, \theta)$, and we give approximation ratios for different families of graphs. We finally show that the S-Labeling problem is polynomial-time solvable for split graphs.

Due to space constraints, proofs are totally absent from this paper. They will be available in its journal version.

1 Preliminaries and Basic Properties

We assume readers have basic knowledge about graph theory [Die00] and we shall thus use most conventional terms of graph theory without defining them (we will only recall basic notations here). Let G be a graph. We write $\mathbf{V}(G)$ for the set of vertices, $\mathbf{E}(G)$ for the set of edges, and $\Delta(G)$ (or Δ, if it is clear from the context) for the maximum degree of G. The *order* (resp. *size*) of G is its number of vertices (resp. edges). The size of a minimum cardinality vertex cover of G is denoted $\tau(G)$. Let G be a graph of order n and size m. A *labeling* of G is a bijective mapping $\theta : \mathbf{V}(G) \to \{1, 2, \ldots, n\}$ and we denote by $\Theta(G)$ the set of all labelings of G. For any graph G and any labeling $\theta \in \Theta(G)$, we let $\mathsf{SL}(G, \theta)$ stand for $\sum_{e \in \mathbf{E}(G)} \min\{\theta(u) : u \in e\}$. To abbreviate notations, we write $\mathsf{SL}(G)$ for $\min\{\mathsf{SL}(G, \theta) : \theta \in \Theta(G)\}$, and $[k]$ for $\{1, 2, \ldots k\}$, where k is a positive integer. We are now in position to formally define the S-Labeling problem we are interested in: Given a graph G of order n, find a bijective mapping $\theta : \mathbf{V}(G) \to [n]$ that minimizes $\mathsf{SL}(G, \theta)$.

Optimal S-labelings of special graphs are informative. It can be easily seen that $\mathsf{SL}(K_n) = \frac{1}{6}n(n^2 - 1)$ is the $(n-1)$-th tetrahedral number. Indeed, $\mathsf{SL}(K_n) = \sum_{1 \leq i \leq n-1} i(n-i) = \sum_{1 \leq i \leq n-1} \sum_{1 \leq j \leq i} j = \sum_{1 \leq i \leq n-1} T_i$, where T_i is the i-th triangular number. Furthermore, $\mathsf{SL}(K_{n,m}) = \frac{nm}{2}(1 + \min\{n, m\})$. Also, $\mathsf{SL}(C_n) = \frac{n^2}{4} + \frac{n}{2} = \mathsf{SL}(P_{n+1})$ if n is even, and $\mathsf{SL}(C_n) = \frac{(n+1)^2}{4} = \mathsf{SL}(P_{n+1})$ if n is odd.

For any graph G and any labeling $\theta \in \Theta(G)$, we write $X(G, \theta)$ for the set of minimum vertices of G with respect to θ, *i.e.*, $X(G, \theta) = \{u : \exists \{u, v\} \in \mathbf{E}(G) \text{ s.t. } \theta(u) < \theta(v)\}$. Straightforward, yet crucial, properties of S-labelings are given in the following two lemmas.

Lemma 1.1 *For any graph G and any labeling $\theta \in \Theta(G)$, $X(G, \theta)$ is a vertex cover of G.*

Lemma 1.2 *For any graph G of order n and size m, there exists a positive integer t, $\tau(G) \leq t < n$, and positive integers $a_i \leq \Delta$, $1 \leq i \leq t$, satisfying (i) $\sum_{1 \leq i \leq t} a_i = m$, (ii) $a_i \geq a_{i+1}$ for any $1 \leq i \leq t-1$, and such that $\mathsf{SL}(G) = \sum_{1 \leq i \leq t} i\, a_i$.*

In the light of Lemma 1.1, it would be tempting to claim that $\tau(G) = |X(G, \theta)|$ for any (or at least one) $\theta \in \Theta(G)$ such that $\mathsf{SL}(G, \theta) = \mathsf{SL}(G)$. Unfortunately, this is not true as shown by considering the $(4, 2)$-bunch graph (4 paths of length 2 having the same end vertex, see Figure 1). The above remark and example raise the question whether $\frac{|X(G,\theta)|}{\tau(G)}$ can be bounded for at least one optimal S-labeling θ of G. We have the following result.

Lemma 1.3 *For any graph G of maximum degree Δ, there exists an optimal S-labeling θ of G such that $\frac{|X(G,\theta)|}{\tau(G)} \leq \sqrt{2\Delta - 1}$.*

Lemma 1.4 *For any graph G of order n and size m,*

$$m\left(\left\lfloor \frac{m}{\Delta} \right\rfloor + 1\right) - \frac{\Delta}{2} \left\lfloor \frac{m}{\Delta} \right\rfloor \left(\left\lfloor \frac{m}{\Delta} \right\rfloor + 1\right) \leq \mathsf{SL}(G) \leq \frac{1}{3} m(n+1)$$

Figure 1. The $(4, 2)$-bunch graph : (a) the Minimum Vertex Cover of cardinality 4 and its associated S-labeling θ such that $\mathsf{SL}(G, \theta) = 20$; (b) an optimal S-labeling θ', achieving a sum of 18, such that $|X(G, \theta')| = 5$. Labels that are not taken into account in the sum are not drawn.

We note that there exists families of graphs for which the upper bound (resp. the lower bound) is reached. Indeed, it can be seen that $\frac{m(n+1)}{3} = \frac{1}{6}n(n^2 - 1) = \mathsf{SL}(K_n)$, since in that case $m = \frac{1}{2}n(n-1)$. Similarly, for any even n, $\mathsf{SL}(C_n) = \frac{n^2+2n}{4} = \frac{\Delta\lfloor\frac{m}{\Delta}\rfloor(\lfloor\frac{m}{\Delta}\rfloor+1)}{2}$, since $\Delta = 2$ and $n = m$ in that case.

However, in the rest of this paper, we shall use another lower bound than the one given in Lemma 1.4 above. More precisely, we will use the following weaker –but simpler– result, which will prove useful in the next section.

Lemma 1.5 *For any graph G of size m and maximum degree Δ,*

$$\mathsf{SL}(G) \geq \frac{m(m + \Delta)}{2\Delta}$$

2 Approximation algorithms

2.1 Deterministic approximation algorithms

In this section, we are interested in giving deterministic approximation algorithms for the S-Labeling problem, for different families of graphs. Since the upper bound from Lemma 1.4 is obtained by the probabilistic method, it cannot be exploited in this context. However, we have the following results.

Lemma 2.1 *For any graph G of order n and size m, there exists a (deterministic) polynomial-time algorithm for computing a labeling θ of G that satisfies $\mathsf{SL}(G, \theta) \leq \frac{mn}{2}$.*

Corollary 2.2 *For any graph G of order n and size m, $\mathsf{SL}(G) \leq \frac{mn}{2}$.*

Thanks to the above corollary and to the lower bound given in Lemma 1.5, we have the following three propositions.

Proposition 2.3 *There exists a polynomial-time deterministic algorithm for the S-*Labeling *problem for regular graphs, whose approximation ratio is* 2.

Proposition 2.4 *There exists a polynomial-time deterministic approximation algorithm for the S-*Labeling *problem for trees of maximum degree Δ, whose approximation ratio is Δ.*

Proposition 2.5 *There exists a polynomial-time deterministic algorithm for the S-*Labeling *problem for graphs of maximum degree Δ, whose approximation ratio is 2Δ.*

2.2 Randomized approximation algorithms

In this section, we are able to provide better approximation ratios than in Propositions 2.3 to 2.5. Indeed, if one does not ask for deterministic algorithms, then the upper bound from Lemma 1.4 can be exploited, and leads to the following three propositions.

Proposition 2.6 *There exists a polynomial-time randomized approximation algorithm for the S-*Labeling *problem for regular graphs, whose expected approximation ratio is $\frac{4}{3}$.*

Proposition 2.7 *There exists a polynomial-time randomized approximation algorithm for the S-*Labeling *problem for trees of maximum degree Δ, whose expected approximation ratio is $\frac{2\Delta}{3}$.*

Proposition 2.8 *There exists a polynomial-time randomized approximation algorithm for the S-*Labeling *problem for graphs of maximum degree Δ, whose expected approximation ratio is $\frac{4\Delta}{3}$.*

3 A polynomial-time algorithm for split graphs

We show in this section that the S-Labeling problem is polynomial-time solvable for *split graphs*. A split graph is a chordal graph with a chordal complement [Gol80]. The vertices of a split graph can be partitioned into a clique K and a stable set S, although this partition may not be unique.

Lemma 3.1 *Let G be a split graph with clique K and stable set S. There exists an optimal S-labeling $\theta \in \Theta(G)$ such that: (i) $1 \leq \theta(u) \leq |K|$ for all $u \in K$, and (ii) for any distinct $u, v \in K$, $\theta(u) < \theta(v)$ if $d_G(u) > d_G(v)$.*

Combining the above lemma with any linear-time recognition algorithm for split graphs [Gol80] (if we assume that the partition $\mathbf{V}(G) = K \cup S$ is not part of the input), we obtain the following result.

Proposition 3.2 *The S-LABELING problem for split graphs is polynomial-time solvable.*

4 Conclusion

In this paper, we have provided a short study of the S-LABELING problem, which was initially proved to be **NP**-complete in [Via06]. We have extracted some basic properties and bounds for optimal S-labelings, before proving different specific results for different classes of graphs. Though this work is still in progress, we would like to point several open problems here, such as: **(1)** What is the complexity of the S-LABELING problem for trees ? for bipartite graphs ?, **(2)** Does there exist a PTAS for the S-LABELING problem, for any graph G ? for specific families of graphs ?, **(3)** Does there exist a constant ratio approximation algorithm for the S-LABELING problem, for any graph G ? for specific families of graphs ? More specifically, is it possible to improve the approximation ratios given in Section 2 ?, and **(4)** Can we obtain a better bound than $\sqrt{2\Delta - 1}$ for $\frac{X(G,\theta)}{\tau(G)}$?

Acknowledgments.

Both authors would like to thank the anonymous referees who, by their remarks, helped improved the contents of this paper.

References

[Die00] R. Diestel, *Graph theory*, second ed., Graduate texts in Mathematics, no. 173, Springer-Verlag, 2000.

[Gol80] M.C. Golumbic, *Algorithmic graph theory and perfect graphs*, Academic Press, New York, 1980.

[Via06] S. Vialette, *Packing of* $(0,1)$*-matrices*, Theoretical Informatics and Applications RAIRO **40** (2006), no. 4, 519–536.