Edge-Disjoint Spanners in Cartesian products of graphs*

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Abstract. A spanning subgraph $S = (V, E')$ of a connected graph $G = (V, E)$ is an $(x + c)$-spanner if for any pair of vertices u and v, $d_S(u, v) \leq d_G(u, v) + c$ where d_G and d_S are the usual distance functions in G and S, respectively. The parameter c is called the delay of the spanner. We study edge-disjoint spanners in graphs, focusing on graphs formed as Cartesian products. Our approach is to construct sets of edge-disjoint spanners in a product based on sets of edge-disjoint spanners and colorings of the component graphs. We present several results on general products and then narrow our focus to hypercubes.

1 Introduction

A spanner of a graph is a spanning subgraph in which the distance between any pair of vertices approximates the distance in the original graph. Although spanners were introduced by Peleg and Ullman [19] for simulation of synchronous distributed systems, they are an interesting graph theoretical structure with application to many problems in interconnection networks [4, 5, 17, 18]. The use of spanners as a network topology (as a substitute for an expensive original topology) was suggested by Richards and Liestman [20] and further studied in a series of papers by Liestman and Shermer [12–16] and Heydemann, Peters, and Sotteau [7]. Algorithms for constructing spanners have also been studied [3, 6, 8, 9].

One problem encountered in parallel computing is to share the resources among several users concurrently. One way to approach this problem is to multitask on

* The research of last three authors was supported by NSERC.

the computers but to dedicate each link to an individual user. In graph-theoretic terms, this corresponds to partitioning the edges into a set of edge-disjoint spanners. Laforest, Liestman, Peleg, Shermer, and Sotteau [11] studied edge-disjoint spanners in complete graphs and in complete digraphs. Laforest, Liestman, Shermer, and Sotteau [10] studied edge-disjoint spanners in complete bipartite graphs. In this paper, we continue this line of study, investigating edge-disjoint spanners in Cartesian products of graphs (and specifically in hypercubes). The remainder of this paper is organized as follows: In section 2, along with other definitions and notation, we define our problem. In section 3, we investigate edge-disjoint spanners in general Cartesian products. In section 4, we restrict our attention to hypercubes.

2 Definitions

A network is represented by a connected simple graph $G = (V(G), E(G))$. We use $d_G(u, v)$ to denote the distance from vertex u to vertex v in graph G. A spanner S of a connected simple graph G is an *$f(x)$-spanner* if for any pair of vertices u and v, $d_S(u, v) \leq f(d_G(u, v))$. We call $d_S(u, v) - d_G(u, v)$ the *delay between vertices u and v in S*. For an $f(x)$-spanner S, we refer to $f(x) - x$ as the *delay* of the spanner. Note that $f(x) - x$ is an upper bound (but not necessarily a tight bound) on the maximum delay in S between any pair of vertices at distance x in G.

We use $H \times G$ to denote the Cartesian product of base graphs H and G. The vertex set $V(H \times G)$ is $V(H) \times V(G) = \{[u, v] : u \in V(H) \text{ and } v \in V(G)\}$. The edge set $E(H \times G)$ contains all pairs $([u, v], [u', v'])$ such that either (1) $u = u'$ and $(v, v') \in E(G)$, or (2) $v = v'$ and $(u, u') \in E(H)$. The following generalization of the Cartesian product is useful in constructing edge-disjoint spanners. Given a coloring of vertices of H, the *color-i product* of graphs H and G, written $H \overset{i}{\times} G$, is the graph with vertex set $V(H) \times V(G)$ and all edges $([u, v], [u', v'])$ such that (1) $u = u'$, the color of u in H is i, and $(v, v') \in E(G)$, or (2) $v = v'$ and $(u, u') \in E(H)$. Note that if all vertices of H are colored i, then the color-i product is simply the Cartesian product. For technical reasons while performing the operation of the color-i product we extend the coloring of H to $H \overset{i}{\times} G$ by assigning the color of $u \in V(H)$ to every vertex $[u, v] \in V(H \overset{i}{\times} G)$.

The (*closed*) *neighborhood* of a vertex v in graph G, denoted $N_G[v]$, is $\{x \in V : d_G(v, x) \leq 1\}$. More generally, the *$d$-neighborhood, $N_G^d[v]$* of v in G is $\{x \in V : d_G(v, x) \leq d\}$.

A *d-dominating set* of vertices in graph G is a set $S \subseteq V$ such that every vertex in V is in the d-neighborhood of some element of S. A *d-domatic coloring* of G is a vertex coloring of G such that each color class constitutes a d-dominating set of G. A d-domatic coloring need not be a proper vertex coloring; we allow adjacent vertices to be assigned the same color. The maximum number of colors in any d-domatic coloring of a fixed graph G is called the *d-domatic number* of G. The 1-domatic number of a graph G is the well-known domatic number of G and will be denoted by $dom(G)$.

Let G be a graph and let $S_1, S_2, \ldots, S_k$ be edge-disjoint subgraphs of G. A vertex coloring of G is called an *all-factor d-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$* if the vertices of each color constitute a d-dominating set in each S_j for $1 \leq j \leq k$. In contrast, a vertex coloring of G with k colors is called a *matched-factor d-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$* if the vertices of each color i constitute a d-dominating set of the subgraph S_i. These colorings were studied by Alon, Fertin, Liestman, Shermer, and Stacho [1] and we will use the results of that paper below.

Our goal is to investigate small delay spanners of Cartesian products. We are particularly interested in those spanners with constant delay, i.e. $(x+c)$-spanners for constant c. More precisely, given a constant c, we are interested in the maximum number of edge-disjoint $(x+c)$-spanners that can be found in G. We let $EDS(G,c)$ denote this number.

3 General Cartesian Products

In this section, we present several results on the number of edge-disjoint spanners that can be found in graphs that are the Cartesian product of other graphs. Typically, these results are lower bounds on the number of spanners in $H \times G$, based on the number of spanners of H and some properties of H or its spanners. We start with a preliminary lemma concerning the delay of a spanner constructed as the Cartesian product of spanners.

Lemma 1 *Let $G_1, G_2, \ldots, G_\alpha$ be graphs and let S_i be a delay c_i spanner of G_i for $i = 1, 2, \ldots, \alpha$. Then $S = S_1 \times S_2 \times \cdots \times S_\alpha$ is a delay c spanner of $G = G_1 \times G_2 \times \cdots \times G_\alpha$, where $c = \sum_{i=1}^{\alpha} c_i$.*

Proof. As each S_i is a spanner of G_i, it follows that S is a spanner of G. Let $u = [u_1, u_2, \ldots, u_\alpha]$ and $v = [v_1, v_2, \ldots, v_\alpha]$ be two vertices of G such that $u_i, v_i \in V(G_i)$ for each i. Then, $d_S(u,v) = \sum_{i=1}^{\alpha} d_{S_i}(u_i, v_i) \leq \sum_{i=1}^{\alpha}(d_{G_i}(u_i, v_i) + c_i) = d_G(u,v) + \sum_{i=1}^{\alpha} c_i$. Thus, S is a delay c spanner of G as claimed.

A central theme in the remainder of this section is the construction of spanners of $H \times G$ that are color products of a spanner H with G, that is, the spanners of $H \times G$ that include the same edges in each copy of H, some entire copies of G, and no other edges. In a particular spanner S, those vertices of H corresponding to a copy of G that is included in S are called *hubs*. We can bound delays of such spanners in $H \times G$ by delays along paths in H that include a hub.

In the following theorem, we use the connectivity of the spanners of H to bound the delay of the spanners of $H \times G$.

Theorem 1 *Let H be a graph on n vertices with k edge-disjoint delay c spanners, each of which is κ-connected. Then for any connected graph G,*

$$EDS\left(H \times G, c + 2\left\lfloor \frac{n - \lfloor n/k \rfloor - 1}{\kappa} \right\rfloor + 2\right) \geq k.$$

Proof. Let $H_1, H_2, \ldots, H_k$ be κ-connected edge-disjoint delay c spanners of H. Color the vertices of H with colors $1, 2, \ldots, k$, in such a way that every color class has cardinality at least $\lfloor \frac{n}{k} \rfloor$ (where we allow adjacent vertices to receive the same color). For $i = 1, 2, \ldots, k$, let $S_i = H_i \overset{i}{\times} G$. These k graphs are edge-disjoint spanners of $H \times G$.

Consider the spanner S_i for some i. Let $u = [h_1, g_1]$ and $v = [h_2, g_2]$ be two vertices of S_i, where $h_1, h_2 \in V(H)$ and $g_1, g_2 \in V(G)$. Let h_1' be a vertex of color i that is closest to h_1 in H_i. The vertex h_1' is a hub of S_i. There is a path P in S_i from $u = [h_1, g_1]$ to $[h_1', g_1]$ to $[h_1', g_2]$ to $[h_1, g_2]$ to $[h_2, g_2] = v$ with length $d_{H_i}(h_1, h_1') + d_G(g_1, g_2) + d_{H_i}(h_1', h_1) + d_{H_i}(h_1, h_2) = 2d_{H_i}(h_1, h_1') + d_G(g_1, g_2) + d_{H_i}(h_1, h_2)$. Since H_i has delay c, $d_{H_i}(h_1, h_2) \leq d_H(h_1, h_2) + c$, and the delay of P is at most $2d_{H_i}(h_1, h_1') + c$.

We now bound $d_{H_i}(h_1, h_1')$. If $h_1 = h_1'$, then $d_{H_i}(h_1, h_1') = 0$, and the lemma follows. Therefore we may assume that $h_1 \neq h_1'$. Let h^* be any vertex of color i in H_i. By our assumption, $h_1 \neq h^*$. Since H_i is κ-connected, there are κ vertex-disjoint paths from h_1 to h^* in H_i. As there are at most $n - 1 - \lfloor n/k \rfloor$ vertices of H_i different from h_1 and of color other than i, one of these vertex-disjoint paths contains at most $\left\lfloor \frac{n-1-\lfloor n/k \rfloor}{\kappa} \right\rfloor$ such vertices. Since h^* has color i, on this path, there must be a vertex of color i at distance at most $\left\lfloor \frac{n-1-\lfloor n/k \rfloor}{\kappa} \right\rfloor + 1$ from h_1. As a consequence, $d_{H_i}(h_1, h_1') \leq \left\lfloor \frac{n-1-\lfloor n/k \rfloor}{\kappa} \right\rfloor + 1$, and the delay of P is at most $2\left\lfloor \frac{n-1-\lfloor n/k \rfloor}{\kappa} \right\rfloor + 2 + c$, giving the result.

By using the same construction but considering the diameters of the spanners of H rather than their connectivities, one obtains the following result.

Theorem 2 *Let H be a graph with k edge-disjoint spanners each of diameter at most d. Then for any connected graph G, $EDS\,(H \times G, 2d) \geq k$.*

In the previous theorems, we have placed relatively few conditions on the coloring used in the color product constructions. In what follows, we make use of more sophisticated colorings to obtain better bounds.

The following theorem uses a natural proper coloring of H and will be a useful starting point for our investigation of hypercube spanners in Section 4.

Theorem 3 *Let H be any bipartite graph and let $c \geq 2$ be an integer. If $EDS(H, c) \geq 2$, then for any connected graph G, $EDS(H \times G, c) \geq 2$.*

Proof. We can properly color vertices of H with colors 1 and 2. Let H_1 and H_2 be edge-disjoint delay c spanners of H. For $i = 1, 2$, let $S_i = H_i \overset{i}{\times} G$. S_1 and S_2 are two edge-disjoint spanners of $H \times G$.

In the following, we bound the delay of S_1, the case of S_2 is similar. Let $u = [h_1, g_1]$ and $v = [h_2, g_2]$ be two vertices of S_1. If $g_1 = g_2$, then both u and v are in some copy of H_1 in $H \times G$ and their delay is at most c. Otherwise $g_1 \neq g_2$.

First consider the case when $h_1 = h_2$. Let h'_1 be a neighbor of h_1 in H_1. Either h_1 or h'_1 is of color 1. If h'_1 has color 1, then there is a path P from $u = [h_1, g_1]$ to $[h'_1, g_1]$ to $[h'_1, g_2]$ to $[h_1, g_2] = v$ with length $1 + d_G(g_1, g_2) + 1 \leq 2 + d_G(g_1, g_2)$. If h_1 is the vertex of color 1, there is a path from u to v with length $d_G(g_1, g_2)$. In either case, the delay is at most $2 \leq c$.

Now consider the case when $h_1 \neq h_2$. If h_1 has color 1, then there is a path in S_1 from $u = [h_1, g_1]$ to $[h_1, g_2]$ to $[h_2, g_2] = v$ with length $d_G(g_1, g_2) + d_{H_1}(h_1, h_2) \leq d_{H \times G}(u, v) + c$. Otherwise, h_1 has color 2. Let h'_1 be a neighbor of h_1 on a shortest path from h_1 to h_2 in H_1. There is a path in S_1 from $u = [h_1, g_1]$ to $[h'_1, g_1]$ to $[h'_1, g_2]$ to $[h_2, g_2] = v$ with length $1 + d_G(g_1, g_2) + (d_{H_1}(h_1, h_2) - 1) \leq d_G(g_1, g_2) + d_H(h_1, h_2) + c = d_{H \times G}(u, v) + c$. Thus, in either case, the delay is at most c.

We have defined matched factor domatic colorings in a way that allows us to construct good spanners using colored products.

Theorem 4 *Let H be a graph with k edge-disjoint delay c spanners $H_1, H_2, \ldots, H_k$. If H has a matched factor l-domatic coloring with respect to $H_1, H_2, \ldots, H_k$, then for any connected graph G, $EDS(H \times G, 2l + c) \geq k$.*

Proof. Consider a matched factor l-domatic coloring of H with respect to $H_1, H_2, \ldots, H_k$ with colors $1, 2, \ldots, k$. For $i = 1, 2, \ldots, k$, let $S_i = H_i \overset{i}{\times} G$. The graphs S_i are edge-disjoint spanners of $H \times G$.

Consider S_i for some i. Let $u = [h_1, g_1]$ and $v = [h_2, g_2]$ be two vertices of S_i. Let h'_1 be a vertex of color i that is closest to h_1 in H_i. The vertex h'_1 is a hub of S_i. There is a path in S_i from $u = [h_1, g_1]$ to $[h'_1, g_1]$ to $[h'_1, g_2]$ to $[h_1, g_2]$ to $[h_2, g_2] = v$, with length $d_{H_i}(h_1, h'_1) + d_G(g_1, g_2) + d_{H_i}(h'_1, h_1) + d_{H_i}(h_1, h_2) = 2d_{H_i}(h_1, h'_1) + d_G(g_1, g_2) + d_{H_i}(h_1, h_2)$. Since we started with a matched factor l-domatic coloring, $d_{H_i}(h'_1, h_1) \leq l$, and the length of this path is at most $2l + d_G(g_1, g_2) + d_H(h_1, h_2) + c = d_{H \times G}(u, v) + 2l + c$.

In [1], with N. Alon, we established that every graph with k edge-disjoint spanners has a matched factor $\lceil \frac{3k-1}{2} \rceil$-domatic coloring. Combining this result with Theorem 4, we obtain:

Corollary 1 *Let H be a graph such that $EDS(H,c) \geq k$, and let G be any graph. Then $EDS(H \times G, 2\left\lceil \frac{3k-1}{2} \right\rceil + c) \geq k$.*

In the previous constructions, we built a set of good spanners in a product from a set of good spanners in a base graph. By using an all-factor domatic coloring, we may build a set of good spanners when one of the base graphs has a single good spanner.

Theorem 5 *Let H be a graph with k edge-disjoint spanners $H_1, H_2, \ldots, H_k$ such that H_1 is a delay c spanner and let H have an all-factor r_1-domatic coloring with k colors with respect to $H_1, H_2, \ldots, H_k$. Let G be a graph with an r_2-domatic coloring with k colors. Then, $EDS(H \times G, 4r_1 + 2r_2 + c) \geq k$.*

Proof. We divide the edges of $H \times G$ into k spanners $S_1, S_2, \ldots, S_k$ as follows: each copy of H in $H \times G$ corresponds to a vertex of G. If this vertex has color i in the domatic coloring of G, we place the edges of $H_1, H_2, \ldots, H_k$ into spanners $S_1, S_2, \ldots, S_k$, respectively, except for H_1 and H_i. We place the edges of H_1 in S_i and the edges of H_i in S_1. For each copy of G in $H \times G$, there is a corresponding vertex of H. If this vertex has color i in the all-factor coloring of H, then we place all edges of this copy of G in S_i.

Let $u = [h_1, g_1]$ and $v = [h_2, g_2]$ be two vertices of S_i. Let h_1' be a vertex of color i that is closest to h_1 in H. Similarly, let h_2' be a vertex of color i that is closest to h_2 in H. Let g_2' be a vertex of color i (in G) that is closest to g_2 in G. There is a path in S_i from $u = [h_1, g_1]$ to $[h_1', g_1]$ to $[h_1', g_2]$ to $[h_1', g_2']$ to $[h_2', g_2']$ to $[h_2', g_2]$ to $[h_2, g_2] = v$; let P be a shortest such path. The length of the subpaths of P from u to $[h_1', g_1]$ and from $[h_2', g_2]$ to v are each at most r_1 by our all-factor r_1-domatic coloring of H. The length of the subpath from $[h_1', g_1]$ to $[h_1', g_2]$ is $d_G(g_1, g_2)$. The length of the subpaths from $[h_1', g_2]$ to $[h_1', g_2']$ and from $[h_2', g_2']$ to $[h_2', g_2]$ are each at most r_2 by our r_2-domatic coloring of G. The length of the subpath from $[h_1', g_2']$ to $[h_2', g_2']$ is $d_{H_i}(h_1', h_2') \leq d_H(h_1', h_2') + c \leq d_H(h_1, h_2) + 2r_1 + c$. Thus, the total distance from u to v along P is $d_G(g_1, g_2) + d_H(h_1, h_2) + 4r_1 + 2r_2 + c = d_{H \times G}(u, v) + 4r_1 + 2r_2 + c$.

With N. Alon, we have shown that any graph with k edge-disjoint spanners has an all-factor $(12k \log k)$-domatic coloring with k colors. (This comes from an exact, rather than asymptotic, analysis of the proof of Theorem 2 in [1].) Combining this with the previous theorem, we obtain:

Corollary 2 *Let H be a graph with k edge-disjoint spanners such that H_1 is a delay c spanner. Let G be a graph with an r-domatic coloring with k colors. Then $EDS(H \times G, 2r + 48k \log k + c) \geq k$.*

In each of the preceding results, we have shown how to construct a set of k spanners in a product graph from a set of at most k spanners in a single base graph. In the following two results, we use domatic colorings and sets of spanners in each base graph to obtain larger set of spanners in the product graph.

Theorem 6 *Let $H_1, H_2, \ldots, H_\alpha$ be graphs. Let $EDS(H_i, c_i) \geq k_i$ for $i = 1, 2, \ldots, \alpha$. If for $i = 1, \ldots, \alpha$, the domatic number $dom(H_i) \geq k_i - \alpha + 1 > 0$, then*

$$EDS(H_1 \times H_2 \times \cdots \times H_\alpha, \sum_{i=1}^{\alpha} c_i + 2 + \max_{i=1,\ldots,\alpha} c_i) \geq \sum_{i=1}^{\alpha} k_i - \alpha^2 + \alpha.$$

Proof. Let $G = H_1 \times H_2 \times \cdots \times H_\alpha$. For $i = 1, 2, \ldots, \alpha$, let $H_{i,1}, H_{i,2}, \ldots, H_{i,k_i}$ be a set of edge-disjoint spanners of H_i each of delay c_i. We will construct $\sum_{i=1}^{\alpha} k_i - \alpha^2 + \alpha$ spanners of G. These spanners are divided into α classes, one class for each H_i. Class i contains $m_i = k_i - \alpha + 1$ spanners $S_{i,1}, S_{i,2}, \ldots, S_{i,m_i}$. Spanner $S_{i,j}$ consists of all copies of $H_{i,j}$ connected by some copies of some $H_{i',j'}$ for every $i' \neq i$. The copies of each spanner $H_{i,m_i+1}, H_{i,m_i+2}, \ldots, H_{i,k_i}$, which are not used in class i, are used to connect all of the spanners of some other class. In particular, for any class i, let

$$R_i = H_{1,t_1+i} \times H_{2,t_2+i} \times \cdots \times H_{i-1,t_{i-1}+i} \times H_{i+1,t_{i+1}+i+1} \times H_{i+2,t_{i+2}+i+1} \times \cdots \times H_{\alpha,t_\alpha+i+1},$$

where $t_i = k_i - \alpha$. In each $S_{i,j}$ some copies of R_i will be used to connect the copies of $H_{i,j}$. To this end, we use a domatic coloring of H_i with colors $1, 2, \ldots, k_i - \alpha + 1$, and let

$$S_{i,j} = H_{i,j} \overset{i}{\times} R_i.$$

As $H_{i,j}$ spans H_i and R_i spans $H_1 \times H_2 \times \cdots \times H_{i-1} \times H_{i+1} \times H_{i+2} \times \cdots \times H_\alpha$, the graph $S_{i,j}$ is a spanner of G.

We now show that all spanners $S_{i,j}$ are edge-disjoint. Consider a pair of spanners $S_{i,j}$ and $S_{i',j'}$. If $i = i'$, then $j \neq j'$. As $H_{i,j}$ and $H_{i,j'}$ are edge-disjoint, and no copy of R_i is in both $S_{i,j}$ and $S_{i,j'}$, (by the color product construction), $S_{i,j}$ and $S_{i,j'}$ are edge-disjoint. Otherwise, $i \neq i'$. In this case, $H_{i,j}$ and $H_{i',j'}$ must be edge-disjoint because H_i and $H_{i'}$ are different spanners. Furthermore, by construction R_i and $R_{i'}$ are also edge-disjoint. Thus, $S_{i,j} \subseteq H_{i,j} \times R_i$ and $S_{i',j'} \subseteq H_{i',j'} \times R_{i'}$ are edge-disjoint.

We now establish the delay of spanner $S_{i,j}$. Let $u = [u_1, u_2, \ldots, u_\alpha]$ and $v = [v_1, v_2, \ldots, v_\alpha]$. Let $u' = [u_1, u_2, \ldots, u_{i-1}, x, u_{i+1}, u_{i+2}, \ldots, u_\alpha]$, where x is a vertex of color j that is closest to u_i in $H_{i,j}$. Let $v' = [v_1, v_2, \ldots, v_{i-1}, x, v_{i+1}, v_{i+2}, \ldots, v_\alpha]$. Let P be a shortest path in $S_{i,j}$ from u to u' to v' to v. The length of the subpath of P from u to u' is at most $c_i + 1$ because u_i and x are at distance 1 in H_i and $H_{i,j}$ is a spanner of H_i with delay c_i. The length of the subpath of P from u' to

v' is at most $\sum_{j\neq i}(d_{H_j}(u_j,v_j)+c_j)$ by Lemma 1. The length of the subpath of P from v' to v is at most $d_{H_i}(u_i,v_i)+1+c_i$ because x and v_i are at distance at most $d_{H_i}(u_i,v_i)+1$ in H_i. Thus, the length of P is at most

$$\begin{aligned}&(c_i+1)+\sum_{j\neq i}(d_{H_j}(u_j,v_j)+c_j)+d_{H_i}(u_i,v_i)+1+c_i\\&=\sum_{j=1}^{\alpha}d_{H_j}(u_j,v_j)+\sum_{j=1}^{\alpha}c_j+c_i+2\\&=d_G(u,v)+\sum_{j=1}^{\alpha}c_j+c_i+2.\end{aligned}$$

Therefore, every $S_{i,j}$ has delay at most $\sum_{j=1}^{\alpha}c_j+\max_{j=1,2,\dots,\alpha}c_j+2$.

The proof of previous theorem can be easily modified so that the following strengthening of Theorem 6 is true. Note that with $m=0$, we get Theorem 6.

Theorem 7 *Let $H_1,H_2,\dots,H_\alpha$ be graphs. Let $0\le m<\alpha$. Let $EDS(H_i,c_i)\ge k_i$ for $i=1,2,\dots,\alpha$. If for $i=1,\dots,\alpha-m$, $dom(H_i)\ge k_i-\alpha+m+1>0$, and for $i=\alpha-m+1,\alpha-m+2,\dots,\alpha$, the value of $k_i\ge\alpha-m-1$, then*

$$EDS(H_1\times H_2\times\cdots\times H_\alpha,\sum_{i=1}^{\alpha}c_i+2+\max_{i=1,\dots,\alpha-1}c_i)\ge\sum_{i=1}^{\alpha-m}k_i-\alpha^2+(2m+1)\alpha-m^2-m.$$

Proof. The proof is similar to the proof of Theorem 6. The difference is that we do not construct classes $\alpha-m+1,\alpha-m+2,\dots,\alpha$, and there are no $R_{\alpha-m+1},R_{\alpha-m+2},\dots,R_\alpha$. Thus, for $i\le\alpha-m$, we need only use $\alpha-m-1$ spanners $H_{i,j}$ to connect spanners of classes other than i (spanner used in $R_1,\dots,R_{\alpha-m}$). This allows us to have $k_i-\alpha+m+1$ spanners in each class i, giving $\sum_{i=1}^{\alpha-m}(k_i-\alpha+m+1)=\sum_{i=1}^{\alpha-m}k_i-\alpha^2+(2m+1)\alpha-m^2-m$ spanners total.

Note that Theorem 7 gives more spanners than Theorem 6 if $\sum_{i=\alpha-m+1}^{\alpha}k_i<(2m+1)\alpha-m^2-m$.

4 Hypercubes

Let Q_d denote the d-dimensional hypercube. Note that $Q_d=Q_{d-i}\times Q_i$ for any $1\le i<d$. In this section, we prove bounds on the number of edge-disjoint spanners that can be found in hypercubes. We begin with two lemmas that show how to construct a set of spanners containing one good spanner. These lemmas will then be combined with Corollary 2 to produce the main results of this section. We use the following result from [21].

Lemma 2 *For any integer $k\ge1$, $dom(Q_{2^k})=2^k$.*

Lemma 3 *For any $k\ge2$ and $d\ge4k-2$, there exists a set of k edge-disjoint spanners $S_1,S_2,\dots,S_k$ of Q_d such that S_1 has delay at most $4k-2$.*

Proof. We express Q_d as the product of two graphs $H = Q_{2k}$ and $G = Q_{d-2k}$. We decompose H into a set of k Hamilton cycles $C_1, C_2, \ldots, C_k$, and arbitrarily choose a distinguished vertex z; see [2]. We construct a 1-domatic coloring of G with k colors. This is trivial for $k = 2$ and possible for $k \geq 3$. Indeed, for such k, we have $d - 2k \geq 2k - 2 \geq 2^{\lceil \log k \rceil} \geq k$. Hence we apply the previous lemma to a $2^{\lceil \log k \rceil}$-dimensional sub-hypercube of G and then extend the $2^{\lceil \log k \rceil}$-coloring to G by adding the remaining dimensions and repeating the coloring in each sub-hypercube. Thus, we obtain the desired 1-domatic coloring of G with k colors (we ignore some color classes if necessary). Further, we decompose G into a set of $k-1$ edge-disjoint spanners $D_2, D_3, \ldots, D_k$. This can be done, for example, by obtaining a set of $k-1$ edge-disjoint Hamilton cycles and dispensing the remaining edges arbitrarily.

Each copy of H in $H \times G$ corresponds to a vertex of a particular color i in the domatic coloring of G. In each copy of H, we place the edges of $C_1, C_2, \ldots, C_k$ into spanners $S_1, S_2, \ldots, S_k$, respectively, except for C_1 and C_i. We place the edges of C_1 in S_i and edges of C_i in S_1. In the copy of G corresponding to the distinguished vertex z, we place the edges of $D_2, D_3, \ldots, D_k$ into spanners $S_2, S_3, \ldots, S_k$, respectively. In every other copy of G, we place all edges into S_1.

Consider a spanner S_i, $2 \leq i \leq k$, and two arbitrary vertices u and v. There is a path from u to v in S_i that starts at u, proceeds within a copy of G to a copy of z, then proceeds within a copy of H to another copy of z, and then proceeds within a copy of G to v. Thus, S_i is connected and, therefore, a spanner.

Now consider a pair of vertices u and v in S_1. If u is a copy of z, let u' be a vertex adjacent to u in S_1 and otherwise let $u' = u$. Similarly, if v is a copy of z, let v' be a vertex adjacent to v in S_1, and otherwise, let $v' = v$. Let $u' = [h_1, g_1]$ and $v' = [h_2, g_2]$. Let $u'' = [h_2, g_1]$, that is u'' is a copy of u' in the copy of H containing v'. We will construct a path from u to v that commences at u, and passes through u', u'', and v' in order and then arrives at v. The subpaths from u to u' and from v' to v are straightforward. The subpath from u'' to v' requires further elucidation. Let H' be the copy of H containing u'' and v', and let z' be the copy of z in H'. Note that the distance from u'' to v' in H' is at most $2k$. If the distance from u'' to v' in H' is at most $2k-2$, then let $u''' = u''$ and $v'' = v'$. If the distance from u''' to v' in H' is $2k-1$, then consider the four edges of S_1 adjacent to u'' or v' in H'. (These edges belong to some cycle C_i in H'.) Of these four edges, two of them (one from u'' and one from v') may be in the dimension of H' in which u'' and v' do not differ. Of the remaining edges, at least one has an end-vertex w other than u'', v', and z'. If w is adjacent to u'' in S_1, then let $u''' = w$ and $v'' = v'$. Otherwise w is adjacent to v' and we let $v'' = w$ and $u''' = u''$. If the distance from u'' to v' in H' is $2k$, then consider the Hamilton cycle formed in H' by S_1. Let w be one of the two vertices at distance two from u'' in this cycle such that $w \neq z'$. Let $u''' = w$ and $v'' = v'$. In all cases above, we have defined u'''

and v'' such that $d_{H'}(u''', v'') \le 2k-2$. Further, we note that the delay between u'' and v' in S_1 is at most the delay between u''' and v'' in S_1.

Now consider a shortest path from u''' to v'' in H'; since either u''' and v'' are adjacent or there are at least two such vertex-disjoint paths, there is a shortest path P that does not contain z'. We construct a path from u''' to v'' in S_1 by replacing each edge of P that is not already in S_1 by a path of three edges in S_1. Let $e = (x, y)$ be an edge of P not in S_1. In H', e belongs to some cycle C_j. In the domatic coloring of G, there is a vertex of color j adjacent to the vertex corresponding to H'. Let x' and y' be the vertices corresponding to x and y, respectively, in the copy of H corresponding to this vertex of color j. By construction, (x', y') is in S_1 and since neither x nor y is z', both (x, x') and (y, y') are edges in S_1. We use the path (x, x', y', y) to replace the edge (x, y) in the path P.

If u and v are both copies of z, then the distance in S_1 from u to u' is 1, the distance in S_1 from u' to u'' is $d_{H\times G}(u, v)$, the distance in S_1 from u'' to v' is at most 6, and the distance in S_1 from v' to v is 1. Thus, in this case, the delay is 8.

If exactly one of the vertices u and v, without loss of generality v, is a copy of z, then the distance in S_1 from u to u' is 0, the distance in S_1 from u' to u'' is $d_G(u, v)$, and the distance in S_1 from v' to v is 1. The path from u''' to v'' in S_1 has length at most $3d_H(u''', v'')$, which is at most $d_H(u''', v'') + 2(2k-2)$ by our choice of u''' and v''. This implies that the distance from u'' to v' in S_1 is at most $d_H(u'', v') + 2(2k-2)$ by our observation that the delay between u'' and v' is at most the delay between u''' and v''. Finally, this quantity is at most $d_H(u, v) + 2(2k-2) + 1$ in this case. This gives a total distance from u to v of at most $d_{H\times G}(u, v) + 2(2k-2) + 2$ and a delay of at most $4k-2$.

If neither u nor v are copies of z, then the distance in S_1 from u to u' is 0, the distance in S_1 from u' to u'' is $d_G(u, v)$, the distance in S_1 from u'' to v' is at most $d_H(u'', v') + 2(2k-2)$ as in the previous case, and the distance in S_1 from v' to v is 0. Since, in this case, $d_H(u'', v') = d_H(u, v)$, this gives a total distance from u to v of at most $d_{H\times G}(u, v) + 2(2k-2)$ and a delay of at most $4k-4$.

For larger d, we can use a similar idea to reduce the delay on S_1 even further.

Lemma 4 *For any $k \ge 2$, $m \ge 2$, and $d \ge 2\binom{m+k-1}{m} + 2(m+k-2)$, there exists a set of k edge-disjoint spanners $S_1, S_2, \ldots, S_k$ of Q_d such that S_1 has delay at most* $\max\{6, 2\left\lceil \frac{2k-1}{m} \right\rceil + 1\}$.

Proof. We proceed as in Lemma 3 expressing Q_d as the product of $H = Q_{2(m+k-1)}$ and $G = Q_{d-2(m+k-1)}$. We decompose H into a set of $m+k-1$ Hamilton cycles and arbitrarily choose a vertex z. We also construct a 1-domatic coloring of G with $\binom{m+k-1}{m}$ colors. Since $d - 2(m+k-1) \ge 2\binom{m+k-1}{m} - 2$, this is possible. As before, we decompose G into $k-1$ edge-disjoint spanners. We can associate each of the $\binom{m+k-1}{m}$ colors with a unique choice of m of the cycles in the decomposition of H.

In the copy of H corresponding to a vertex of color χ of G, we place the edges of the cycles associated with χ into S_1. The remaining $k-1$ cycles in this copy of H are each placed into one of the $k-1$ spanners $S_2, S_3, \ldots, S_k$. The edges of the copies of G are placed into the spanners as in the proof of Lemma 3.

The analysis of the delay of S_1 is quite similar to the proof of Lemma 3 and we only point out the major differences. To construct a path from u'' to v' (two vertices in the same copy H' of H), we begin by taking as many edges of S_1 as possible in the direction of v' and not leading to z'. When no such further step is possible, we are at vertex u''' such that $d_{H'}(u''', v') \leq 2k-1$. Consider a shortest path P from u''' to v' in H' that does not contain z'. We divide P into $\left\lceil \frac{|V(P)|-1}{m} \right\rceil$ subpaths of length at most m. Each such subpath has edges from at most m of the cycles in the decomposition of H. Thus, by the domatic coloring, there is a copy H'' of H adjacent to H' in which all of the edges of this subpath are in spanner S_1. We replace this subpath with an edge to H'', the corresponding subpath in H'', and an edge back to H', encountering 2 units of delay. Since $length(P) \leq 2k-1$, there are at most $\left\lceil \frac{2k-1}{m} \right\rceil$ subpaths giving delay at most $2\left\lceil \frac{2k-1}{m} \right\rceil$. We get delay at most $2\left\lceil \frac{2k-1}{m} \right\rceil + 1$ when one of u and v is a copy of z, and delay 6 when both are.

Note that the usable limit of this lemma is when $m = k$ giving a spanner of delay 6 in Q_d, where $d = 2\binom{2k-1}{k} + 4k - 4$.

As promised, we now combine the previous lemmas with Corollary 2 to give the main results of this section. We use $G = Q_{2k-2}$ (which has a 1-domatic coloring with k colors) in Corollary 2, and $H = Q_{d-2k+2}$ from Lemma 3 or 4 to obtain Theorem 8 and Theorem 9, respectively.

Theorem 8 *For $k \geq 2$ and $d \geq 6k-4$, $EDS(Q_d, 48k \log k + 4k) \geq k$.*

Theorem 9 *For $k \geq 2$, $m \geq 2$, and $d \geq \binom{m+k-1}{m} + 2m + 4k - 4$,*

$$EDS(Q_d, 48k \log k + 2 + \max\{6, 2\left\lceil \frac{2k-1}{m} \right\rceil + 1\}) \geq k.$$

These theorems show that one can find k edge-disjoint spanners with delay $O(k \log k)$ in Q_d for sufficiently large d. In particular, beyond a certain dimension, the delay depends only on the number of spanners and not the size of the cube.

Lemma 5 *$EDS(Q_4, 4) = 2$.*

Proof. Let us consider the following decomposition of Q_4 into two Hamilton cycles, see Figure 1. One of the Hamilton cycles is depicted in bold edges and the another

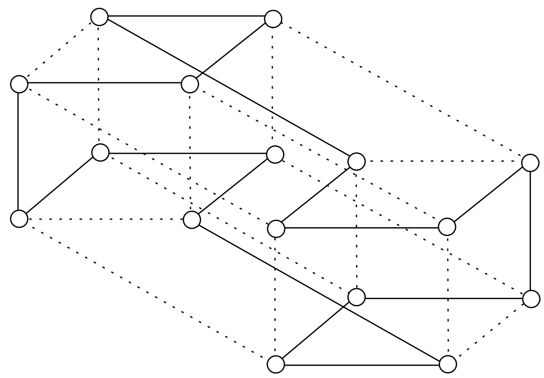

Fig. 1. Two Hamilton cycles in Q_4.

in dotted edges. It is only a time consuming exercise to check that both these Hamilton cycles are delay 4 spanners in Q_4. Since every spanner of Q_4 must have at least 15 edges, and Q_4 has only 32 edges, $EDS(Q_4, 4) \leq 2$.

Theorem 10 *For $d \geq 6$, $EDS(Q_d, 18) \geq 3$.*

Proof. For brevity, we here outline the general method of constructing three spanners of delay 18 in Q_6. The full details of the construction are given in the Appendix. After the sketch of the proof for Q_6, we describe how to extend the construction to higher dimensions.

We view Q_6 as four copies H_1, H_2, H_3, H_4 of Q_4 interconnected by four sets of edges. We decompose each H_i into a Hamilton cycle and two matchings. Spanners S_1 and S_2 contain Hamilton cycles in H_1 and H_2, respectively, and each contains three matchings, one each in the remaining H_i's. Spanner S_3 includes the remaining two Hamilton cycles and two matchings.

Each of the four sets of interconnecting edges is divided in half. To do this, we 2-color each Q_4 in the same manner. An interconnecting edge is placed in one subset if its ends are colored 1 and placed in the other subset otherwise. The two subsets of each set of interconnecting edges will be assigned to two different spanners. In particular, we give the edges between H_1 and H_2 to S_1 and S_2, the edges between H_2 and H_3 to S_2 and S_3, the edges between H_3 and H_4 to S_1 and S_2, and the edges between H_4 and H_1 to S_1 and S_3, see Figure 2.

At this point, S_1 and S_2 are connected and each contains a single cycle. The subgraph S_3, however, consists of two components, each with one cycle. To ensure that all three subgraphs are spanners, we may exchange one or more of the interconnecting edges of S_1 between H_3 and H_4 with an equal number of edges of S_3 in the Hamilton cycle of H_4. At this point, each S_i is connected and contains one cycle. In particular, each S_i contains the Hamilton cycle in H_i.

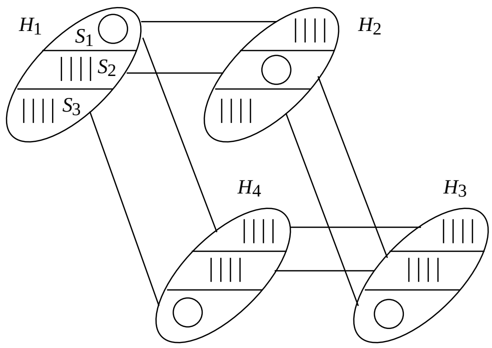

Fig. 2. Q_6 expressed as four copies of Q_4.

If we ignore the delays introduced by the edges exchanged between S_1 and S_3, we can easily obtain a rough estimate of the delay between any two vertices u and v in S_1 or S_2. There is a path from u to v in S_i $(i = 1, 2)$ that consists of three sections: one from u to the cycle in H_i, one around the cycle, and one from the cycle to v. To get from u to the cycle takes at most four edges, going around the cycle takes at most half the cycle length (eight edges), and to get from the cycle to v takes at most four more edges. This is a total distance of at most 16, a delay at most 14 between u and v. The actual analysis of S_1 and S_3 must take into account the exchanged edges. This analysis is tedious and contains no insight and is thus omitted here. By careful choice of the decomposition of each H_i, which matchings to assign to each spanner, which subsets of interconnecting edges to assign to each spanner, and which edges to exchange, we may obtain a set of spanners with maximum delay 18. This construction has been verified by computer and the details of the construction are given in the appendix.

To extend this construction to higher dimensions, we start with the three spanners S_1, S_2, and S_3 in Q_6 as described above. Again, we view Q_6 as four copies of Q_4, and we color the vertices of H_1 with color 1, H_2 with color 2, and $H_3 \cup H_4$ with color 3. To construct three spanners S_1', S_2', and S_3' of Q_d for $d > 6$, we group the lower 6 dimensions and the upper $d-6$ dimensions and view Q_d as $Q_6 \times Q_{d-6}$. We then let each S_i' be the color-i product $S_i \overset{i}{\times} Q_{d-6}$. We now establish that each of the spanners S_1', S_2', and S_3' has delay at most 18. Consider two vertices u and v in S_i'. If u and v do not differ in any of the upper $d-6$ dimensions, then the delay between them is at most 18 by the construction above.

If u and v differ in the upper dimensions, consider a shortest path P from u to v' where v' has the same lower coordinates as v and upper coordinates as u. If P includes a vertex w of color i, then we may construct a path from u to v by following P from u to w, following edges in the upper dimensions as necessary, and then following the remainder of P projected to the copy of Q_6 containing v.

Since there is no delay encountered in traveling the upper dimensions, this path has delay at most 18.

If P does not include a vertex of color i, then let w be the closest vertex of color i to u. Observe that w is also the closest vertex of color i to v'. Based on the sketch above, w is within distance 6 of both u and v'. For the exact construction presented in the appendix, these distances are at most 5. The path from u to w followed by the necessary upper dimension edges to a vertex w' and then to v has delay at most 12.

Lemma 6 *For* $d \geq 2$, $EDS(Q_d, 2^{d-1} - 2) = \lfloor \frac{d}{2} \rfloor$.

Proof. This follows from the fact that $\lfloor \frac{d}{2} \rfloor$ Hamilton cycles can be found in Q_d, and the fact that the delay of any spanner of any bipartite graph must be even.

4	0	4										
5	0	4										
6	0	4	18									
7	0	4	18									
8	0	4	18	126								
9	0	4	18	138								
10	0	4	18	138	510							
11	0	4	18	138	524							
12	0	4	18	56	524	2046						
13	0	4	18	56	524	2064						
14	0	4	18	56	272	2064	8190					
15	0	4	18	56	272	2064	8210					
16	0	4	18	56	272	380	8210	32766				
17	0	4	18	56	272	398	8210	32790				
18	0	4	18	56	132	398	1148	32790	131070			
19	0	4	18	56	132	398	1160	32790	131096			
20	0	4	18	56	132	310	1160	1532	131096	524286		
21	0	4	18	56	132	310	1160	1556	131096	524316		
22	0	4	18	56	132	310	672	1556	4604	524316	2097150	
23	0	4	18	56	132	310	672	1556	4618	524316	2097182	
24	0	4	18	56	132	170	672	888	4618	6140	2097182	8388606
25	0	4	18	56	132	170	672	900	4618	6170	2097182	8388642
d/k	1	2	3	4	5	6	7	8	9	10	11	12

Table 1. Upper bounds on the delay of k spanners in Q_d for small k and d.

We conclude this section with Table 1 which shows a lower bound on the delay for a set of k spanners in Q_d. These bounds were obtained by the application of various results from this paper. For each value entered in Table 1, the corresponding entry in Table 1$'$ indicates how the value was obtained. The bottom line indicates particular set of edge-disjoint spanners used. For example, the string aQb, cQd indicates a spanners of Q_b and c spanners of Q_d.

Table 1′. Used results to get the corresponding entries in Table 1.

d/k	1	2	3	4	5	6	7	8	9	10	11	12
4	Trivial	Lem. 5										
5	Trivial	Thm. 3										
6	Trivial	Thm. 3	Thm. 10									
7	Trivial	Thm. 3	Thm. 10									
8	Trivial	Thm. 3	Thm. 10	Lem. 6								
9	Trivial	Thm. 3	Thm. 10	Cor. 1 4Q8								
10	Trivial	Thm. 3	Thm. 10	Cor. 1 4Q8	Lem. 6							
11	Trivial	Thm. 3	Thm. 10	Cor. 1 4Q8	Cor. 1 5Q10							
12	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q6	Cor. 1 5Q10	Lem. 6						
13	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q7	Cor. 1 5Q10	Cor. 1 6Q12						
14	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q8	Thm. 6 3Q6, 4Q8	Cor. 1 6Q12	Lem. 6					
15	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q9	Thm. 6 3Q7, 4Q8	Cor. 1 6Q12	Cor. 1 7Q14					
16	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q10	Thm. 6 3Q8, 4Q8	Thm. 6 4Q8, 4Q8	Cor. 1 7Q14	Lem. 6				
17	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q11	Thm. 6 4Q8, 3Q9	Cor. 1 6Q16	Cor. 1 7Q14	Cor. 1 8Q16				
18	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q12	Thm. 6 3Q6, 4Q12	Cor. 1 6Q16	Thm. 6 4Q8, 5Q10	Cor. 1 8Q16	Lem. 6			
19	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q13	Thm. 6 3Q6, 4Q13	Cor. 1 6Q16	Thm. 6 4Q9, 5Q10	Cor. 1 8Q16	Cor. 1 9Q18			
20	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q14	Thm. 6 3Q6, 4Q14	Thm. 6 4Q8, 4Q12	Thm. 6 4Q10, 5Q10	Thm. 6 5Q10, 5Q10	Cor. 1 9Q18	Lem. 6		
21	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q15	Thm. 6 3Q6, 4Q15	Thm. 6 4Q8, 4Q13	Thm. 6 5Q10, 4Q11	Cor. 1 8Q20	Cor. 1 9Q18	Cor. 1 10Q20		
22	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q16	Thm. 6 3Q6, 4Q16	Thm. 6 4Q8, 4Q14	Thm. 6 4Q8, 5Q14	Cor. 1 8Q20	Thm. 6 5Q10, 6Q12	Cor. 1 10Q20	Lem. 6	
23	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q17	Thm. 6 3Q6, 4Q17	Thm. 6 4Q8, 4Q15	Thm. 6 4Q8, 5Q15	Cor. 1 8Q20	Thm. 6 5Q11, 6Q12	Cor. 1 10Q20	Cor. 1 11Q22	
24	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q18	Thm. 6 3Q6, 4Q18	Thm. 6 4Q12, 4Q12	Thm. 6 4Q8, 5Q16	Thm. 6 4Q8, 6Q16	Thm. 6 5Q12, 6Q12	Thm. 6 6Q12, 6Q12	Cor. 1 11Q22	Lem. 6
25	Trivial	Thm. 3	Thm. 10	Thm. 6 3Q6, 3Q19	Thm. 6 3Q6, 4Q19	Thm. 6 4Q12, 4Q13	Thm. 6 4Q8, 5Q17	Thm. 6 4Q9, 6Q16	Thm. 6 6Q12, 5Q13	Cor. 1 10Q24	Cor. 1 11Q22	Cor. 1 12Q24

5 Appendix

In this appendix, we include a description of a set of three edge-disjoint spanners of Q_6. These spanners have delay 14, 14, and 18. We view the set of spanners as a 3-coloring of edges and describe the construction in Table 2 by listing the label of every vertex along with the colors of its incident edges in order of increasing dimension i. For example, vertex 000000 has edges in dimension 4 and 5 in spanner S_1, in dimensions 1 and 3 in spanner S_2, and in dimension 2 and 6 in spanner S_3. The delay of each of these spanners has been verified by computer.

000000	2 3 2 1 1 3	010000	3 2 1 1 1 2	100000	3 3 2 1 2 3	110000	1 3 2 2 2 2
000001	2 2 3 1 3 1	010001	3 1 2 1 3 1	100001	3 2 3 1 3 1	110001	1 2 3 2 3 1
000010	3 3 2 1 3 1	010010	3 2 1 1 3 1	100010	3 3 2 1 3 1	110010	1 3 2 2 3 1
000011	3 2 1 3 1 2	010011	3 1 1 2 1 2	100011	3 2 1 3 2 2	110011	1 2 2 3 2 2
000100	3 3 2 1 3 1	010100	3 2 1 1 3 1	100100	3 3 2 1 3 1	110100	1 3 2 2 3 1
000101	3 2 3 1 1 2	010101	3 1 2 1 1 2	100101	3 2 3 1 2 2	110101	1 2 3 2 2 2
000110	2 3 2 1 1 3	010110	3 2 1 1 1 2	100110	3 3 2 1 2 3	110110	1 3 2 2 2 2
000111	2 2 1 3 3 1	010111	3 1 1 2 3 1	100111	3 2 1 3 3 1	110111	1 2 2 3 3 1
001000	3 2 3 1 3 1	011000	3 1 2 1 3 1	101000	3 2 3 1 3 1	111000	1 2 3 2 3 1
001001	3 2 3 1 1 2	011001	3 1 2 1 1 2	101001	3 2 3 1 2 2	111001	1 2 3 2 2 2
001010	2 2 3 1 1 3	011010	3 1 2 1 1 2	101010	3 2 3 1 2 3	111010	1 2 3 2 2 2
001011	2 2 1 3 3 1	011011	3 1 1 2 3 1	101011	3 2 1 3 3 1	111011	1 2 2 3 3 1
001100	2 2 3 1 1 3	011100	1 3 2 1 1 2	101100	2 3 3 1 2 3	111100	2 1 3 2 2 2
001101	2 3 3 1 3 1	011101	1 3 2 1 3 1	101101	2 3 3 1 3 1	111101	2 1 3 2 3 1
001110	2 2 3 1 3 1	011110	1 3 2 1 3 1	101110	2 3 3 1 3 1	111110	2 1 3 2 3 1
001111	2 3 1 3 1 2	011111	1 3 1 2 1 2	101111	2 3 1 3 2 2	111111	2 1 2 3 2 2

Table 2. Representation of three spanners in Q_6.

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