FACTOR d-DOMATIC COLORINGS OF GRAPHS

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ABSTRACT. Consider a graph and a collection of (not necessarily edge-disjoint) connected spanning subgraphs (factors) of the graph. We consider the problem of coloring the vertices of the graph so that each color class of the vertices dominates each factor. We find upper and lower bounds on $\alpha(t, k)$, which we define as the minimum radius of domination d such that every graph with a collection of k factors can be vertex colored with t colors so that each color class d-dominates each factor. It is perhaps surprising that the upper bound is finite and does not depend on the order of the graph. We obtain similar results for a variant of the problem where the number of colors is equal to the number of factors and each color class must d-dominate only the corresponding factor rather than all factors.

1. INTRODUCTION

Factor domination was introduced by Brigham and Dutton [2]. In their paper, a decomposition of a graph into k edge-disjoint spanning factors is called a *k-factoring.* A subset of the vertices of the graph is a *factor dominating set* if it is a dominating set for each of the factors. Brigham and Dutton studied the minimum cardinality of factor dominating sets. A similar concept, global domination, was introduced by Sampathkumar [9]. A summary of results in this area appears in a paper by Brigham and Carrington [1].

A comprehensive survey of results on domination can be found in the books of Haynes, Hedetniemi, and Slater [6, 7]. Following the work on distance domination, a natural generalization of factor domination is to allow the factor dominating set to be a d-dominating set for each of the factors rather than a 1-dominating set (For a survey of results on distance domination the reader is referred to [8]). Rather than simply studying the cardinality of these factor d-dominating sets, one could study the equivalent notion of the domatic number: how many such sets can be packed into the graph?

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In this paper, we study this domatic number problem from a slightly different angle. First, we will require that each of the factors is connected. Second, rather than fixing d and finding the maximum number of factor d-dominating sets, we fix the number of such sets and find the minimum d for that number of sets. Third, we do not require the factors to be edge-disjoint. Allowing edges to appear in more than one factor is necessary for some of the applications mentioned by Brigham and Carrington [1]. We also investigate a variation of this problem in which the number of sets is equal to the number of factors and each set is required to d-dominate a single corresponding factor rather than all factors.

We now introduce some definitions in order to formalize these notions. Let $G = (V, E)$ denote a connected graph with vertex set V and edge set E. By a *factor* of G, we mean any connected spanning subgraph of G. We use the term *k-factorization* of G to denote a set of k factors $S_1, S_2, \ldots, S_k$ of G whose union is G. (Note that usually factorization implies that the factors are edge-disjoint.) The k-factorization of G can also be viewed as a multi-coloring of the edges with k colors.

We use $d_G(u, v)$ to denote the distance from vertex u to vertex v in graph G. The *neighborhood* of a vertex v in graph G is $\{x \in V : d_G(v, x) \leq 1\}$. More generally, the *$d$-neighborhood* of v in G is $\{x \in V : d_G(v, x) \leq d\}$.

A *d-dominating set* of vertices in graph G is a set $S \subseteq V$ such that every vertex in V is in the d-neighborhood of some element of S. A *d-domatic coloring* of G is an assignment of colors to the vertices of G such that each color class constitutes a d-dominating set of G. Note that a d-domatic coloring (and all colorings in this paper) need not be a proper vertex coloring; we allow adjacent vertices to be assigned the same color. The maximum number of colors in any d-domatic coloring of a fixed graph G is called the *d-domatic number* of G. Our 1-domatic number is known more commonly as simply the *domatic number*.

Let G be a graph and let $S_1, S_2, \ldots, S_k$ be a k-factorization of G. A vertex coloring of G is called an *all-factor d-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$* if the vertices of each color constitute a d-dominating set in each S_j for $1 \leq j \leq k$. Note that an all-factor d-domatic coloring of G is also a d-domatic coloring of G. In contrast, a vertex coloring of G with k colors is called a *matched-factor d-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$* if the vertices of each color i constitute a d-dominating set of the factor S_i. Note that a matched-factor d-domatic coloring of G is again a d-domatic coloring of G.

Given integers t and k, we are interested in the minimum $d(t, k)$ such that every k-factorization of every graph on at least t vertices admits an all-factor $d(t, k)$-domatic coloring with t colors. We let $\alpha(t,k)$ denote this minimum $d(t,k)$. Similarly, given an integer k, we let $\mu(k)$ denote the minimum $d(k)$ such that every k-factorization of every graph on at least k vertices admits a matched-factor $d(k)$-domatic coloring. The notations α and μ are intended to be mnemonic, indicating all-factor and matched-factor, respectively.

The focus of this paper is to establish bounds on $\alpha(t,k)$ and on $\mu(k)$. For α, we give two upper bounds and a lower bound. For μ, we obtain more precise bounds, showing that $k \le \mu(k) \le \lceil 3(k-1)/2 \rceil$, when $k \ge 2$.

Our interest in these problems was incited by an application of matched-factor d-domatic colorings to the construction of spanners in cross products of graphs [4].

2. All-factor d-domatic Colorings

We begin with a lemma that we will use in all of the upper bounds.

Lemma 1. *For any tree T on vertex set V, $|V| = n \ge k \ge 1$, there exists a partition of V into $V_1, V_2, \ldots, V_p$ such that for every $1 \le i \le p$, V_i contains a subset B_i such that $|B_i| = k$, and for every $u \in V_i$ and $v \in B_i$, $d_T(u,v) \le \lceil 3(k-1)/2 \rceil$.*

Proof. The proof is by induction on the diameter of T. In the base step we consider all trees with the diameter at most $2k-2$. Let r be a vertex from the center of T, and let u and v be two vertices of maximum distance in T chosen such that $d_T(r,v) \le d_T(r,u)$. For any vertex $x \in V$, it holds that $d_T(r,x) \le d_T(r,u) \le k-1$. We let $V_1 = V$, and $B_1^* = \{x : x \in V_1 \text{ and } d_T(r,x) \le \lceil (k-1)/2 \rceil\}$. We claim that $|B_1^*| \ge k$. This is obviously true if $d_T(r,u) \le \lceil (k-1)/2 \rceil$. Otherwise, since $d_T(r,v) \ge d_T(r,u) - 1$, the subset B_1^* contains $\lceil (k-1)/2 \rceil$ vertices ($\neq r$) of the unique $r-u$ path, and $\lceil (k-1)/2 \rceil$ vertices ($\neq r$) of the unique $r-v$ path. This together with the vertex r guarantees at least k vertices in B_1^*. The distance from any vertex $y \in V_1$ to any vertex $z \in B_1^*$ is at most $d_T(y,r) + d_T(r,z) \le k-1+\lceil (k-1)/2 \rceil = \lceil 3(k-1)/2 \rceil$. Let B_1 be any size-k subset of B_1^*. Thus V_1 with B_1 is a partition satisfying the lemma.

Now assume that the diameter of T is at least $2k-1$. It follows that T contains an edge e such that both subtrees T_1 and T_2 of $T-e$ have at least k vertices and both T_1 and T_2 have diameter smaller than T. Hence, we can apply induction and partition the vertex set of T_1 into subsets $V_1, V_2, \ldots, V_m$ (and corresponding B_i's) with the desired properties, and the vertex set of T_2 into subsets $V_{m+1}, V_{m+2}, \ldots, V_p$ (and corresponding B_i's) with the desired properties. It follows that $V_1, V_2, \ldots, V_p$ with $B_1, B_2, \ldots, B_p$ satisfy the lemma. □

The upper bound in the previous lemma is tight as shown by the path of length $2k-2$.

In the following theorem, we combine Lemma 1 with Hall's theorem [5] to obtain a general upper bound on $\alpha(t,k)$.

Theorem 1. *For every $k \geq 2$ and $t \geq k$, $\alpha(t,k) \leq \lceil 3(kt-1)/2 \rceil$.*

Proof. Let $S_1, S_2, \ldots, S_k$ be a k-factorization of a graph G on $n \geq t$ vertices. If $n < kt$, each S_i has diameter at most $kt-2 \leq \lceil 3(kt-1)/2 \rceil$, and any coloring with t colors will work. Now consider the case when $n \geq kt$. For $i = 1, 2, \ldots, k$, let T_i be a spanning subtree of S_i. By Lemma 1, each T_i, and hence each S_i, can have its vertices partitioned into p_i subsets $V_{i,1}, V_{i,2}, \ldots, V_{i,p_i}$ where each $V_{i,j}$ $(1 \leq j \leq p_i)$ contains $B_{i,j}$ such that $|B_{i,j}| = kt$ and for every $x \in V_{i,j}$ and $y \in B_{i,j}$, $d_{S_i}(x,y) \leq \lceil 3(kt-1)/2 \rceil$. For every $1 \leq i \leq k$ and $1 \leq j \leq p_i$, partition the set $B_{i,j}$ into t blocks (subsets) $B_{i,j,l}$ $(l = 1, 2, \ldots, t)$ each of cardinality k. Now we can define the bipartite graph H with bipartition B and W as follows. The class B contains all the blocks $B_{i,j,l}$, $1 \leq i \leq k$, $1 \leq j \leq p_i$, and $1 \leq l \leq t$. The class W contains all the vertices of G. Both classes have at most n elements. The edges of H are defined as follows. Each $B_{i,j,l} \in B$ is joined to a vertex $v \in W$, if and only if $v \in B_{i,j,l}$. Obviously, $deg_H(B_{i,j,l}) = k$, for each $B_{i,j,l} \in B$, and $deg_H(v) \leq k$, for each $v \in W$. So we can use Hall's theorem to find a matching that saturates the partition B.

Now, for each i $(1 \leq i \leq k)$ and for each j $(1 \leq j \leq p_i)$ we color the vertices matched to $B_{i,j,l}$ $(1 \leq l \leq t)$ with color l. All remaining vertices of G are colored by any of the t colors. It follows from the properties of the blocks that the coloring is an all-factor $\lceil 3(kt-1)/2 \rceil$-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$. Hence, $\alpha(t,k) \leq \lceil 3(kt-1)/2 \rceil$. □

When k is large, we can obtain a better bound using a probabilistic argument; we use the following symmetric version of Lovász' Local Lemma [3].

Lemma 2. *(Lovász' Local Lemma) Let $A_1, A_2, \ldots, A_n$ be events in a probability space. Let H be a graph with vertices $A_1, A_2, \ldots, A_n$ such that for each $i = 1, 2, \ldots, n$, the event A_i is independent of any combination of events that are not neighbors of A_i in H. Suppose that the maximum degree of H is d, the probability $P[A_i] \leq p$ for all i, and that $4dp < 1$. Then $P[\wedge_{i=1,\ldots,n} \overline{A_i}] > 0$.*

In the proof of the following theorem, we use the notation $\log x$ to denote the base-2 logarithm of x; we follow the same convention throughout the remainder of the paper.

Theorem 2. *For every $k \geq 1$ and $t \geq 1$, $\alpha(t,k) \leq O(t \cdot \log(kt))$.*

Proof. Let $S_1, S_2, \ldots, S_k$ be a k-factorization of $G = (V, E)$, where $|V| \geq t$. Randomly color the vertices of G with colors $1, 2, \ldots, t$ such that $P[col(v) = i] = 1/t$ $(i = 1, 2, \ldots, t)$ for any vertex $v \in V$. If $|V| < 4t\log(kt)$, then any coloring using all t colors will be an all-factor $O(t \cdot \log(kt))$-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$. Otherwise $|V| \geq 4t\log(kt)$, and by Lemma 1, each spanning subtree of S_i, and hence each S_i, can have vertices partitioned into p_i subsets $V_{i,1}, V_{i,2}, \ldots, V_{i,p_i}$ where each $V_{i,j}$ $(1 \leq j \leq p_i)$ contains a subset $B_{i,j}$ of size $4t\log(kt)$ and for every $x \in V_{i,j}$ and $y \in B_{i,j}$, $d_{S_i}(x,y) \leq 6t\log(kt)$.

For every subset $B_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq p_i$, define the event $A_{i,j}$ that some color is missing on the vertices of $B_{i,j}$. It is not difficult to see that $\overline{A_{1,1}} \wedge \ldots \wedge \overline{A_{1,p_1}} \wedge \overline{A_{2,1}} \ldots \wedge \overline{A_{k,p_k}}$ is the event that G has an all-factor $6t\log(kt)$-domatic coloring with respect to $S_1, S_2, \ldots, S_k$. To prove that $P[\overline{A_{1,1}} \wedge \ldots \wedge \overline{A_{1,p_1}} \wedge \overline{A_{2,1}} \ldots \wedge \overline{A_{k,p_k}}] > 0$ we will make use of Lovász' Local Lemma. Define a graph H as follows. Vertices of H are all events $A_{i,j}$. Two vertices/events of H are adjacent if the corresponding vertex sets have non-empty intersection. Clearly, each event $A_{i,j}$ is independent of events which are not its neighbors in H.

Let us estimate the maximum degree in H. For fixed i, every vertex appears in at most one vertex set corresponding to an event of the form $A_{i,j}$, $1 \leq j \leq p_i$. Since each event corresponds to a set of $4t\log(kt)$ vertices, it follows that the maximum degree in H is at most $4t\log(kt)(k-1)$. Finally, it is not hard to observe that $P[A_{i,j}] \leq t(1-1/t)^{|B_{i,j}|} \leq t/(kt)^4$.

The inequality in Lovász' Local Lemma is implied by $16t^2(k-1)\log(kt)/(kt)^4 < 1$, which is satisfied for all $k \geq 1$ and $t \geq 1$. Hence Lemma 2 guarantees that $P[\overline{A_{1,1}} \wedge \ldots \wedge \overline{A_{1,r_1}} \wedge \overline{A_{2,1}} \ldots \wedge \overline{A_{k,p_k}}] > 0$. This in turn means that there exists an all-factor $6t\log(kt)$-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$. Hence $\alpha(t,k) \leq O(t \cdot \log(kt))$. □

Corollary 1. *For $k \geq 1$, $\alpha(k,k) \leq O(k\log k)$.*

Since $\mu(k) \leq \alpha(k,k)$, the above corollary provides a simple upper bound on $\mu(k)$. We can also show a lower bound on $\alpha(t,k)$ when $t \geq 2$ and $k \geq 4$.

Lemma 3. *For every integer $k \geq 4$ and $p \geq 1$, let $n = p \cdot \lfloor \log k \rfloor$. Then there are k permutations of $1, 2, \ldots, n$ such that every set of $\left\lfloor \frac{1}{2} \lfloor \log k \rfloor \right\rfloor$ elements appears in the left half of at least one permutation.*

Proof. If $p = 1$, then we have $n = \lfloor \log k \rfloor$. It follows that $k \geq 2^n > \binom{n}{\lfloor n/2 \rfloor}$, and we can choose k permutations such that every set of $\lfloor \frac{1}{2} \lfloor \log k \rfloor \rfloor = \lfloor \frac{n}{2} \rfloor$ elements appear in the left half of at least one of them, as claimed.

If $p > 1$, then we have $n > \lfloor \log k \rfloor$. We can split $1, 2 \ldots, n$ into $\lfloor \log k \rfloor$ blocks of size p. Now we choose k permutations of $1, 2, \ldots, \lfloor \log k \rfloor$ as in the first case applying these permutations to the blocks of size p. These k permutations have the property that every set of $\lfloor \frac{1}{2} \lfloor \log k \rfloor \rfloor$ blocks appears in the left half of at least one permutation. By replacing each block with its corresponding elements in these permutations, we obtain k permutations of $1, 2, \ldots, n$ for which every set of $\lfloor \frac{1}{2} \lfloor \log k \rfloor \rfloor$ elements appears in the left half of at least one permutation. Indeed, every element belongs to exactly one block, and thus $\lfloor \frac{1}{2} \lfloor \log k \rfloor \rfloor$ elements appear in at most $\lfloor \frac{1}{2} \lfloor \log k \rfloor \rfloor$ blocks. □

Note that the restriction $k \geq 4$ in the previous lemma is required in order to ensure that $\lfloor \frac{1}{2} \lfloor \log k \rfloor \rfloor \geq 1$.

Theorem 3. *If $t \geq 2$ and $k \geq 4$, then $\alpha(t, k) \geq \Omega(t \cdot \log k)$.*

Proof. Since $\alpha(t, k) \geq \alpha(t - 1, k)$, we may assume that t is even in order to show an asymptotic lower bound. We will construct a graph on $n = \frac{1}{2} t \cdot \lfloor \log k \rfloor \geq \lfloor \log k \rfloor$ vertices. Let S be a collection of k Hamiltonian paths on these vertices chosen so that for each set of $\lfloor \frac{1}{2} \lfloor \log k \rfloor \rfloor$ vertices there is a Hamiltonian path in S in which they all appear in the left half. The existence of S is guaranteed by Lemma 3.

Let G be the union of all the Hamiltonian paths in the collection S, and note that the collection forms a factorization of G. Let ϕ be an all-factor d-domatic coloring of G with respect to this factorization, where d is as small as possible. By the Pigeonhole principle, in any coloring of G by t colors, at least one color class, say i, does not appear more than $\lfloor \frac{n}{t} \rfloor = \lfloor \frac{1}{2} \lfloor \log k \rfloor \rfloor$ times. Consider a Hamiltonian path of S in which all the vertices of color i appear in the left half. In this path, the distance from any vertex of color i to the rightmost vertex of the path is at least $\lfloor \frac{1}{4} t \cdot \lfloor \log k \rfloor \rfloor$. Since $k \geq 4$, $n = \frac{t}{2} \lfloor \log k \rfloor \geq t$ showing that the graph G is in the domain of $\alpha(t, k)$. The theorem follows. □

3. Matched-factor d-domatic Colorings

As noted above, Corollary 1 provides an upper bound of $O(k \log k)$ on $\mu(k)$. Again, by applying Lemma 1 and Hall's theorem, we can substantially improve on this bound.

Theorem 4. *For every $k \geq 2$, $\mu(k) \leq \lceil 3(k - 1)/2 \rceil$.*

Proof. Let $S_1, S_2, \ldots, S_k$ be a k-factorization of a graph G on $n \geq k$ vertices. We may assume that they are trees. By Lemma 1, each S_i can be partitioned into p_i subsets $V_{i,1}, V_{i,2}, \ldots, V_{i,p_i}$ where each $V_{i,j}$ $(1 \leq j \leq p_i)$ contains a subset $B_{i,j}$ such that $|B_{i,j}| = k$ and for every $x \in V_{i,j}$ and $y \in B_{i,j}$, $d_{S_i}(x, y) \leq \lceil 3(k-1)/2 \rceil$.

Construct the bipartite graph H with bipartition B and W as follows. The class B contains all the subsets $B_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq p_i$. The class W contains all the vertices of G. Both classes have at most n elements. The edges of H are defined as follows. Each $B_{i,j} \in B$ is joined to a vertex $v \in W$, if and only if $v \in B_{i,j}$. Obviously, $deg_H(B_{i,j}) = k$, for each $B_{i,j} \in B$, and $deg_H(v) \leq k$, for each $v \in W$. So we can use Hall's theorem to find a matching that saturates the partition B.

Now, for each i $(1 \leq i \leq k)$ we color the vertices matched to $B_{i,j}$ $(1 \leq j \leq p_i)$ with color i. All remaining vertices of G are colored by any of the k colors. It follows from the properties of the blocks that the coloring is a matched-factor $\lceil 3(k-1)/2 \rceil$-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$. Hence, $\mu(k) \leq \lceil 3(k-1)/2 \rceil$. □

The lower bound $\mu(k) \geq k - 1$ can be established by considering the graph G that is simply a path on k vertices, with each of the k factors equal to G. As our final result, we improve this bound by one. Our proof has the added feature that the collection of factors that we construct is edge-disjoint.

Theorem 5. *For all $k \geq 2$, $\mu(k) \geq k$.*

Proof. We exhibit a graph G with a decomposition $S_1, S_2, \ldots, S_k$ into edge-disjoint factors such that G has no matched-factor $(k-1)$-domatic coloring with respect to $S_1, S_2, \ldots, S_k$. Each of $S_1, S_2, \ldots, S_k$ is a tree. Since $S_1, S_2, \ldots, S_k$ are edge-disjoint, we will speak of an edge of S_i as having color i.

The graph G consists of k subgraphs $H_1, H_2, \ldots, H_k$ all having a common vertex a, and some extra connecting structure. Each H_i is designed so that its decomposition prohibits a from having color i in a matched-factor $(k-1)$-domatic coloring.

We will now describe the construction of the subgraph H_1. The construction of any other H_i is identical except that the colors are permuted so that color i is mapped to color 1 in H_1. The subgraph H_1 has $k(k-1)+2$ vertices, including the shared vertex a, another distinguished vertex b_1, and a set $T = \{v_{x,y} : 0 \leq x \leq k-1 \text{ and } 0 \leq y \leq k-2\}$. The vertices a and b_1 are the only vertices of H_1 that have edges to vertices outside of H_1. Inside of H_1, the edges of each color j, $2 \leq j \leq k$, form a tree consisting of k paths of length $k-1$ with common vertex a, using all of the vertices of H_1 except b_1. Also inside H_1, the edges of color 1 form a tree consisting of $k-1$ paths of length k with common vertex b_1, using all of the vertices of H_1 except a.

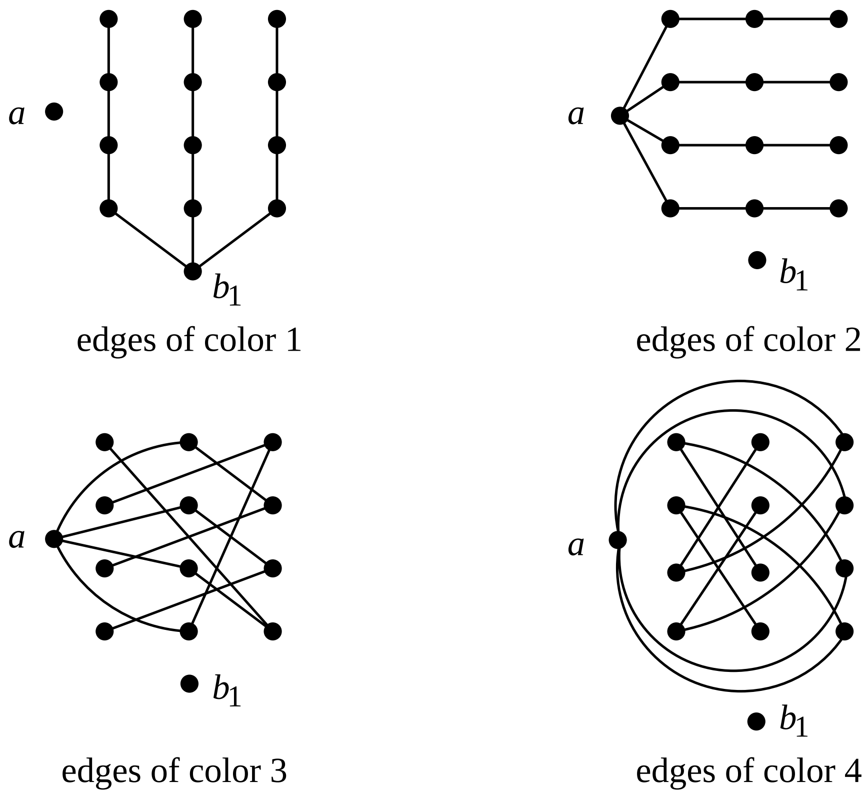

FIGURE 1. Case $k = 4$ in the proof of Theorem 5

Before proceeding with the details of how to achieve this structure, we show that if we can achieve this structure, then the theorem follows.

Suppose, by way of contradiction, that a has color 1 in such a coloring. Consider the tree of color i in H_1. Each leaf of this tree must be within distance $k-1$ of a vertex of color i in S_i. Since a has color 1 and a and b_1 are the only vertices connecting H_1 with the rest of G, it follows that each leaf is dominated by a distinct vertex of color i. Thus, there are at least k vertices of each color $2, 3, \ldots, k$ in T. This accounts for all of the vertices of T and so no vertex of T is colored 1. However, this is a contradiction as, even if b_1 is colored 1, the leaves of the color 1 tree have no vertex of color 1 within distance $k-1$ in S_1. Thus, H_1 ensures that a may not have color 1 in a matched-factor $(k-1)$-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$.

Similarly, each H_i ensures that a may not have color i. It follows that there is no matched-factor $(k-1)$-domatic coloring of G with respect to $S_1, S_2, \ldots, S_k$.

It remains to show that we can construct $H_1, H_2, \ldots, H_k$, and G as described above. Again, rather than giving the construction for all H_i, we give the construction only for H_1.

The edges of color 1 are $\{b_1 v_{k-1,y} : 0 \le y \le k-2\} \cup \{v_{x,y} v_{x+1,y} : 0 \le x \le k-2, 0 \le y \le k-2\}$. The edges of color j, $2 \le j \le k$ are $\{a v_{x,j-2} : 0 \le x \le k-1\} \cup \{v_{x,y} v_{x+(j-2) \pmod{k}, (y+1) \pmod{k-1}} : 0 \le x \le k-1, 0 \le y \le k-2, y \ne (j-3) \pmod{k-1}\}$.

It is easy to check that these edge colorings constitute edge-disjoint forests as described above.

In order to connect each forest of given edge color into a tree, we must add some additional structure. In particular, add $k-1$ new vertices $c_1, c_2, \ldots, c_{k-1}$, and k edge-disjoint paths (one in each S_i) connecting the vertices $\{a, b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_{k-1}\}$. For $k \ge 3$, this is possible because K_{2k} can be decomposed into k Hamiltonian cycles. For $k = 2$, K_4 can be decomposed into two Hamiltonian paths. This completes the proof. □

Trivially, $\mu(1) = 0$. Theorem 4 and Theorem 5 provide exact values for $\mu(2)$ and $\mu(3)$.

Corollary 2. $\mu(2) = 2$ *and* $\mu(3) = 3$.

4. Conclusion

We defined all-factor and matched-factor d-domatic colorings of graphs with factorizations. We introduced $\alpha(t,k)$ to denote the minimum d such that every graph with a factorization into k factors has an all-factor d-domatic vertex coloring with t colors. Similarly, we introduced $\mu(k)$ to denote the minimum d such that every graph with a factorization into k factors has a matched-factor d-domatic vertex coloring with k colors. We established upper and lower bounds on both α and μ. Surprisingly, these bounds do not depend on the number of vertices in the graph.

Our bounds for μ are fairly close but our bounds for α are not. Tightening these bounds remains an open problem.

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