Diameter of Knödel Graph

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Abstract

We show that the diameter of 2^k-nodes Knödel graph of degree k is $\lceil (k+2)/2 \rceil$.

1 Introduction

Recently, diverse properties and invariants of interconnection networks, not only those of parallel machines, have been studied and a number of interesting results has been found, see e.g. [1]. One of the important features of an interconnection network is its message passing ability. The quality of an interconnection network depends mainly on the time delay of the communication between the nodes, which can be either processors or computers or other type terminals. Let us suppose that a node knows a piece of information and needs to transmit it to every other node in the network. This task is usually called broadcasting. When each node knows a piece of information that has to be transmitted to every other node, we are speaking about gossiping. Gossiping can be viewed as simultaneous broadcasting. We refer to [6, 7] for good surveys on broadcasting and gossiping.

The Knödel graph was introduced 25 years ago [9] as an interconnection network where gossiping could be performed in the minimum time if the 1-port model is supposed, that is, at any given time a node can exchange information at most with one of its neighbouring nodes. More precisely, Knödel gave an algorithm in a fully-connected network i.e. in a complete graph, so that it allows gossiping to be achieved in minimum time and Fraigniaud and Peters [5] have formally defined the Knödel graph, which is the graph underlying Knödel's construction.

The Knödel graph of 2^k nodes and degree k contains the minimum possible number of links (edges) while the time needed to achieve gossiping is the same as it would be in the complete graph. In other words, this graph is a *minimum gossip graph*. It can also be

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shown that the Knödel graph of 2^knodes and degree k is a minimum broadcast graph. In [4], it was shown that many of the graphs given as examples of minimum broadcast graphs in [3, 8] or of minimum gossip graphs [10], not only for $n = 2^k$, are in fact isomorphic to the Knödel graph.

Let us define the Knödel graph in general: The Knödel graph on $n \geq 2$ nodes, n is even, and of maximum degree $1 \leq k \leq \lfloor \log_2(n) \rfloor$, denoted $W_{k,n}$, is composed of nodes (i, j) where $i = 1, 2$ and $0 \leq j \leq n/2 - 1$ and of edges between any two nodes: $(1, j)$ and $(2, j + 2^t - 1 (\bmod n/2))$ for any $j, 0 \leq j \leq n/2 - 1$ and $t, 0 \leq t \leq k - 1$; the edges connecting the nodes: $(1, j)$ and $(2, j + 2^t - 1 (\bmod n/2))$ for any $j, 0 \leq j \leq n/2 - 1$ are said to be in dimension t.

Diameter is one of the main invariants of an interconnection network giving the number of hops in a shortest path that a message has to perform on its way from the source to the destination node in the worst case. Although the Knödel graph was introduced 25 years ago, the diameter of the Knödel graph has not been determined.

In this paper we show that the diameter of the Knödel graph of 2^k nodes and degree k is $\lceil (k + 2)/2 \rceil$.

In the following section we introduce an alternative definition of the Knödel graph which will be useful for the following considerations. In section 3 we show the upper bound on the diameter and in section 4 we show the lower bound. The last section is devoted to some open questions and conclusions.

2 Alternative definition

We introduce an alternative definition of the Knödel graph $W_{k,n}$, where $n = 2^k$. Let us take the Hamiltonian cycle of the Knödel graph composed of edges of dimensions 0 and 1 and label the nodes of the Knödel graph as follows: $(1, j) = 2j, (2, j) = 2j - 1 (\bmod n), 0 \leq j \leq n/2 - 1$. It means that a node of the Knödel graph is labelled by a $k-$bit number and each node, labelled by an even number x, is incident to the edges connecting it to the nodes: $x + 2^t - 3 (\bmod n)$, for each $1 \leq t \leq k$. The edges corresponding to a t are the edges of the $(t - 1)$th dimension of the general definition of the Knödel graph. See figure 1 with the Knödel graph $W_{4,16}$.

As the Knödel graph is vertex symmetric (this fact follows easily from the original definition of the Knödel graph), to determine the diameter, it is sufficient to study the distance from the node 0 to any node x. If there is a path from the node 0 to a node x using f edges of dimensions: $t_1, t_2, t_3, ..., t_f$, it means that

$$x = \sum_{i=1}^{f} (-1)^{i-1} (2^{t_i+1} - 3) (\bmod n).$$

Therefore, determining the shortest path between 0 and x is equivalent to finding the above representation of x with minimal possible f. The representation will be called $\pm$representation and we will say that the length of the $\pm$representation is f.

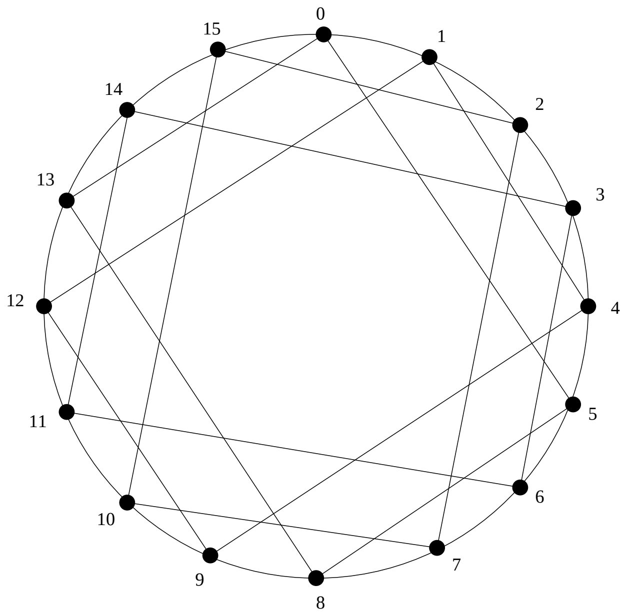

Figure 1: Knödel graph $W_{4,16}$

3 Upper bound

Consider the binary representation of a label x of a node in the Knödel graph:

$$x = \sum_{t=0}^{k-1} l_t 2^t$$

where $l_t \in \{0, 1\}, t = 0, 1, 2, ..., k-1$. First we consider the even case.

3.1 Even case

Assume that we create a $\pm$representation of x in the following form, hereinafter, we will call this representation the basic $\pm$representation.

Let x be an even $k-$bit number. Then

$$x = \sum_{t=1}^{m} (2^{i_t} - 2^{j_t}),$$

such that $k \geq i_1 > j_1 > i_2 > j_2 > ... > i_m > j_m \geq 1$, where each pair $2^{i_t} - 2^{j_t}$ represents the block of ones in the binary representation of x i.e. $2^{i_t} - 2^{j_t} = 2^{i_t-1} + 2^{i_t-2} + ... + 2^{j_t}$. In the Knödel graph each pair $2^{i_t} - 2^{j_t}$ represents two edges: one edge from the i_t-th dimension and the other from the j_t-th dimension which follows after the first one and the $\pm$representation gives a path how the node x can be reached from the node 0.

Let $I = \{i_1, i_2, ..., i_m\}$, $J = \{j_1, j_2, ..., j_m\}$.
Let $\#(A)$ denote the cardinality of a set A.
A maximal sequence of consecutive ones in the binary representation of x is called a block.
A maximal set of consecutive numbers from $I \cup J$ is called a $\pm$block.

A number a such that $1 \le a \le k$ and $a \notin I \cup J$ we will call a gap or a gap number if more suitable. Notice that $0 \notin I \cup J$ and we will not count it to gap numbers. An important fact is that each ±block is accompanied with one gap number which is a number by one greater than the maximum of the block. The only exception is a (possible) block with k. Let B_e^+ be the set of ±blocks of integers from $I \cup J$ such that the maximums of the ±blocks are from I and the minimums of the ±blocks are from J i.e. the length of a ±block from B_e^+ is even. Let B_e^- be the set of even length ±blocks of integers from $I \cup J$ with maximums from J. Similarly, we define the set of ±blocks B_o^+ (B_o^-) which is the set of the ±blocks with maximums from I (J) and the lengths of the blocks are odd and strictly greater than 1. For the sets of blocks of length 1 we will use notation B_1^+ and B_1^-. The sum of lengths of all ±blocks in a set of ±blocks B_p^*, where $p \in \{e, o\}$, and $* \in \{+, -\}$ will be called total length of the set and denoted $|B_p^*|$.

The following proposition shows that we can find a ±representation of x such that its length is less or equal to $\lceil (k+2)/2 \rceil$.

Proposition 3.1 *For the basic ±representation of x it holds that $\#(I \cup J) \le \lceil (k+2)/2 \rceil$ or we can transform the basic ±representation into a ±representation, where new I and J are sets (not multisets) s.t. $\#(I \cup J) \le \lceil (k+2)/2 \rceil$.*

Proof. First we introduce four reduction rules and then we show some useful lemmas.

Let us define four reduction rules R1-R4.

Reduction rule R1
if $a \neq b \in I$ s.t. $a-1, b+1 \in J$, exist **then**
do $I = I \cup \{a-1\} - \{a, b\}, J = J \cup \{b\} - \{a-1, b+1\}$ **od**

Reduction rule R2
if $a, b, c \in I$ are pairwise not equal s.t. $a-1, b-1, c+2 \in J, a-1 \neq c+2 \neq b-1$, exist and $c+1 \notin I \cup J$ **then**
do $I = I \cup \{a-1, b-1\} - \{a, b, c\}, J = J \cup \{c, c+1\} - \{a-1, b-1, c+2\}$ **od**

The reduction rule R3 is symmetrical to the rule R2.

Reduction rule R3
if $a, b, c \in I$ are pairwise not equal s.t. $a+1, b+1, c-2 \in J, a+1 \neq c-2 \neq b+1$, exist and $c-1 \notin I \cup J$ **then**
do $I = I \cup \{c-1, c-2\} - \{a, b, c\}, J = J \cup \{a, b\} - \{a+1, b+1, c-2\}$ **od**

Reduction rule R4
if $k, a \in I, k \neq a, a-1 \in J$ **then**
do $I = I \cup \{a-1\} - \{k, a\}, J = J - \{a-1\}$

The correctness of the rules follows from the equivalences:
R1 : $2^a + 2^b - 2^{a-1} - 2^{b+1} = 2^{a-1} - 2^b$
R2 : $2^a + 2^b + 2^c - 2^{a-1} - 2^{b-1} - 2^{c+2} = 2^{a-1} + 2^{b-1} - 2^{c+1} - 2^c$
R3 : $2^a + 2^b + 2^c - 2^{a+1} - 2^{b+1} - 2^{c-2} = 2^{c-1} + 2^{c-2} - 2^a - 2^b$

R4 : $2^k + 2^a - 2^{a-1} (\mathrm{mod}\ 2^k) = 2^{a-1}$.
Therefore the rules R1-R4, if applied to a ±representation of x, produce a new ±representation of x.

Note that any application of the reduction rules R1 and R4 creates two new gap numbers $a, b+1$ and k, a, respectively. Any application of the reduction rule R2 introduces three new gap numbers $a, b, c+2$ while the original gap number $c+1$ is changed to a number from J i.e. by applying the rule R2 the number of gaps in the new ±representation is increased by 2. Similar situation holds for the rule R3, where three new gap numbers $a+1, b+1, c$ are created and the original gap number $c-1$ is changed to the number from I. Again, the number of gap numbers in the new ±representation is increased by 2.

Now, we describe the application of the above rules on a basic representation.
The following lemma shows a procedure how to get a new ±representation where only one of the set of blocks $B_e^+, B_e^-, B_o^+, B_o^-$ remains non empty and possibly $B_1^+ \neq \emptyset$, $B_1^- \neq \emptyset$.

Lemma 3.1 *Let $B_e^+ \neq \emptyset$ and $B_e^- \neq \emptyset$. We can reduce both sets of ±blocks to a new set of ±blocks of the same type as it was the type of the set of greater length. Moreover we get a set of ±blocks of length 1 each accompanied with a gap number. A similar proposition holds for all following combinations: $B_o^+ \neq \emptyset$ and $B_o^- \neq \emptyset$; $B_e^+ \neq \emptyset$ and $B_o^+ \neq \emptyset$; $B_e^+ \neq \emptyset$ and $B_o^- \neq \emptyset$; $B_e^- \neq \emptyset$ and $B_o^+ \neq \emptyset$; $B_e^- \neq \emptyset$ and $B_o^- \neq \emptyset$.*

Proof of lemma 3.1 Let $B_e^+ \neq \emptyset$ and $B_e^- \neq \emptyset$. Let us apply the reduction rule R1, as many times as possible so that we take a from a ±block of B_e^+ and b from a ±block of B_e^- and a and b are the maximal not yet in R1 used numbers of I and J, respectively. We get new blocks of length 1. Furthermore, if in the original ±representation $|B_e^+| = |B_e^-|$ then in the new ±representation $B_e^+ = B_e^- = \emptyset$ and if $|B_e^+| > |B_e^-|$ ($|B_e^+| < |B_e^-|$) we get a new ±representation where $B_e^+ \neq \emptyset$ and $B_e^- = \emptyset$ ($B_e^+ = \emptyset$ and $B_e^- \neq \emptyset$). There may be created at most one new ±block which is of odd length, greater than 1. Let the maximum of the block be d. Then $d, d-1 \in I$ ($d, d-1 \in J$ if $|B_e^+| < |B_e^-|$) and the part of the ±block except d is a part of the original ±block from B_e^+. This new ±block is accompanied with two gap numbers - the gap number accompanying the original ±block and the gap number $d+1$. Therefore, we can consider the new ±block to be an even length ±block with maximum $d-1$ and accompanied with one gap number. Formally we can consider d as a ±block from B_1^+ accompanied with a gap number too. □

One can see that the application of rule R1 affects each element of $I \cup J$ of the basic ±representation at most once and the gap numbers of the basic ±representation are not affected at all. For the rest we need to process the still not affected part of the ±representation. We discuss two cases $B_o^+ \neq \emptyset$ and $B_e^+ \neq \emptyset$ in the following two lemmata.

Lemma 3.2 *If in a ± representation $B_o^+ \neq \emptyset, B_e^+ = B_e^- = B_o^- = \emptyset$ then we can create a new ±representation where the number of gaps is at least $\#(I \cup J) - 3$.*

Proof of the lemma 3.2. Let $B_o^+ \neq \emptyset, B_e^+ = B_e^- = B_o^- = \emptyset$. We can suppose that B_o^+ does not contain any ±block of length $4i+1, i \geq 1$. Otherwise, we can apply the reduction rule R1 so that the $a, a-1, b, b+1$ (see the rule R1) are taken from the same ±block of the length $4i+1$. If we do it to maximal possible extent, we get for each such ±block (except the one with k as a maximum) as many gaps as the number of elements in new I and J. We can also suppose that B_o^+ does not contain more than one ±block of length 3. Otherwise we can efficiently use the reduction R1 on the set of all the ±blocks of length 3. If the number of the ±blocks is odd, then one such ±block remains.

It remains to analyse the case where all ±blocks of B_o^+ are of length $4i+3, i \geq 1$.

On each ±block of B_o^+ we can use the reduction rule R1. We can do it in a way that after reduction we get for each such ±block $u, u-1, u-3 \in I$ and $u-2 \in J$. With each such new ±block two gap numbers are accompanied: $u+1 \notin I \cup J$ and the one which was accompanied with the original ±block (except the case when the maximum of the original ±block is equal to k.) If we get two such ±blocks i.e. u, v exist such that: $u, u-1, u-3 \in I, u-2 \in J, v, v-1, v-3 \in I, v-2 \in J$ (it holds: $\{u, u-1, u-2, u-3\} \cap \{v, v-1, v-2, v-3\} = \emptyset$), then we can apply the reduction rule R1 taking for example $u-1, u-2$ from one ±block and $v-2, v-3$ from the other ±block. Create as many as possible pairs of type u, v and apply R1 on all the pairs. After the latest reduction we get at least $\#(I \cup J) - 1$ gaps except in the case where the number of ±blocks in B_o^+ was odd.

Now, let us analyze the odd case.

Let $k \notin I$. By applying the same procedure, as it was used for the even case, we get $\#(I \cup J) - 2$ gaps.

Let $k \in I$. If $k \in I$ and the number of ±blocks is more than 1, then apply R4 so that a is a maximum of another ±block. Now we may have an even length ±block with maximum $k-1 \in J$ and one even length ±block of form: $a-1, a-2, a-3, a-4, ..., v+1, v$ s.t. $a-1, a-2, a-4, ..., v \in I, a-3, a-5, ..., v+1 \in J$ and $a+1, a, v-1$ are gap numbers. If the length of ±block with maximum $k-1$ is shorter than the total length of the other ±blocks−1, then we can apply the reduction rule R1 so that in place of the ±block with maximum $k-1$ we will get as many gaps as the numbers from J, and this case is reduced to the previous one. We will get at least $\#(I \cup J) - 2$ gaps again.

Let the length of the ±block with maximum $k-1$ be greater than the total length of the other ±blocks−1. Then we can use the reduction rule R1 again so that one couple (in the rule shown as $b+1, b$) is taken from the ±block with maximum $k-1$ and the other one is taken from other ±blocks. In the place of other ±blocks we get the same number of gaps as $\#(I)$ and in the place of the original ±block with maximum $k-1$ we get a ±block of form $u, u-1, u-3, ..., v-3, v-1 \in J, u-2, u-4, u-6, ..., v-2, v \in I$ and we apply the rule R1 to it. Finally, two ±blocks may rest: either $d, d-1 \in J$ and $g, g-1 \in I$, then the number of gaps is at most by one less than $\#(I \cup J)$ or $d, d-1, d-3 \in J$ and $d-2, g, g+1 \in I$ and we have at least $\#(I \cup J) - 3$ gaps and we are done.

Let the basic ±representation contains only one ±block. Then the following reduction will take place: apply R4 so that $a = k-2$ and then apply R1 to the rest of the ±block. In the worst case $k = 4i+5$ and there are $2i+1$ gaps and we are done. □

Lemma 3.3 *If in a ± representation $B_e^+ \neq \emptyset, B_e^- = B_o^+ = B_o^- = \emptyset$ then we can create a new ±representation where the number of gaps is at least $\#(I \cup J) - 3$.*

Proof of the lemma 3.3. Let $B_e^+ \neq \emptyset, B_e^- = B_o^+ = B_o^- = \emptyset$. If $k \in I$, apply R4 so that a is a maximum of any other ±block. We get an odd length ±block $B = k-1, k-2, ..., y$ s.t. $k-1, k-3, k-5, ..., y \in J$ and $k-2, k-4, k-6, ..., y+1 \in I$ and one odd length ±block of form: $u-1, u-2, u-3, u-4, ..., v+1, v$ s.t. $u-1, u-2, u-4, ..., v \in I, u-3, u-5, ..., v+1 \in J$ and $y-1, u+1, u, v-1$ are gap numbers. Now we can use the rule R1 so that $a = k-1, a-1 = k-2$ and as $b, b+1$ can be taken from any other ±block

- the only condition is that we take it from the shortest one and from its beginning. If the $\pm$block B is exhausted by the rule R1 or it was empty, we continue as it is in the case where $k \notin I \cup J$ (see below).

If the length of the $\pm$block B is greater than the total length of the other $\pm$blocks (here we do not count the $\pm$blocks of length 1) , then in the place of other $\pm$blocks we get as many gaps as non gap numbers (we also count the original gap numbers). We can apply to the rest of the $\pm$block B the reduction rule R1 so that the $b+1, b$ of the rule is taken from the beginning of the rest and $a, a-1$ from the end of the rest of the $\pm$block B. Either we get a $\pm$block $y, y-1 \in J$ or a $\pm$block $y, y-1, y-2, y-3$ where $y, y-1, y-3 \in J, y-2 \in I$. In the first case, there are as many gap numbers as non gap numbers in place of the original $\pm$block with the maximum k and we are done. In the second case, the number of gap numbers is smaller by two than the number of the non gap numbers in the place of the original $\pm$block with maximum k and we are done.

Let $k \notin I$. Let us use the reduction rule R1 in such a way that $a, a-1$ we take from the shorter $\pm$blocks and $b+1, b$ i we take from the longer $\pm$blocks.
If one $\pm$block remains, then we apply the rule so that $a, a-1$ is from the beginning of the $\pm$block and $b+1, b$ from its end. In the worst case, a $\pm$block remains composed of: $a, a-1, a-3 \in J$ and $a-2 \in I$ such that $a+1, a-4 \notin I \cup J$ or a $\pm$block composed of: $a, a-1, a-3 \in I$ and $a-2 \in J$ such that $a+1, a-4 \notin I \cup J$. As there are at least $\#(I \cup J) - 3$ gaps, we are done.
If two same length $\pm$blocks remain, then we apply the reduction rule R1 at the maximal possible extent (if the length is 2, we do not apply it) so that we take $a, a-1$ from the first of the blocks and $b+1, b$ from the second of the blocks. In this case there are at least $\#(I \cup J) - 3$ gaps in the new $\pm$representation.
If three same length $\pm$blocks remain, then we apply R2, if possible, so that we take c from the end of one of the $\pm$blocks and then we apply R1 similarly as in the previous case. If rule R2 is impossible to apply then there are at least 2 gaps behind each of at least two of these $\pm$blocks (it means that behind these two $\pm$blocks are in the basic $\pm$representation $\pm$blocks which are associated with 2 gaps - not only 1). By application of R1 at maximal possible extent we get (one should consider the above mentioned 2 superficial gaps from the basic $\pm$representation) a $\pm$representation with at least $\#(I \cup J)$ gaps.
If four $\pm$blocks remain, we can apply R2, if possible, and R1 in the similar way as in the case of three $\pm$blocks or we can argue similarly if R2 can not be used so that for these four $\pm$blocks the number of gaps is the same as the numbers in $I \cup J$. It means that in the new $\pm$representation we have at least $\#(I \cup J) - 1$ gaps.
Finally, if $4j + i, 0 \le i \le 3$, $\pm$blocks remain, we can create a new $\pm$representation with at least $\#(I \cup J) - 3$ gaps so that we apply R2 and R1 j times in the way as for four blocks and then for the rest i blocks we apply the previous cases accordingly. □

The cases: $B_o^- \neq \emptyset, B_e^+ = B_e^- = B_o^+ = \emptyset$ and $B_e^- \neq \emptyset, B_e^+ = B_o^+ = B_o^- = \emptyset$ are symmetrical (we have to use R3 rule in place of R2) to the cases discussed in the Lemma 3.2 and in the Lemma 3.3, respectively.

In the resulting $\pm$representation, if $k = 2l$ then it holds $\#(I \cup J) \le l + 1$ and if $k = 2l + 1$ then it holds $\#(I \cup J) \le l + 2$. In both cases we have $\#(I \cup J) \le \lceil (k+2)/2 \rceil$. □

3.2 Odd case

Let x be an odd k-bit number. The basic $\pm$representation of x is created as follows:

Let the three least significant bits of the binary representation of x be 001. Then the basic $\pm$ representation of x is defined as follows:

$$x = \sum_{t=1}^{m}(2^{i_t} - 2^{j_t}) + 2^2 - 3,$$

where $k \geq i_1 > j_1 > i_2 > j_2 > ... > i_m > j_m \geq 3$. We determine a new $\pm$representation so that we use the same procedure as in the case of x even to the $k-2$ most significant bits of the binary representation of x. Such a $\pm$representation is at most $\lceil k/2 \rceil$ long. This $\pm$representation completed by $2^2 - 3$ is a new $\pm$representation of x s.t. its length is at most $\lceil k/2 \rceil + 1 = \lceil (k+2)/2 \rceil$.

Let the three least significant bits of the binary representation of x be 101. Then the basic $\pm$represention of x is as follows:

$$x = \sum_{t=1}^{m}(2^{i_t} - 2^{j_t}) + 2^3 - 3,$$

where $k \geq i_1 > j_1 > i_2 > j_2 > ... > i_m > j_m \geq 4$. According to the previous subsection we can find a $\pm$representation of the $k-3$ most significant bits of the binary representation of x s.t. its length is at most $\lceil (k-1)/2 \rceil$. This $\pm$representation together with $2^3 - 3$ is a new $\pm$representation of x s.t. its length is at most $\lceil (k-1)/2 \rceil + 1 < \lceil (k+2)/2 \rceil$ as k is odd.

If the three least significant bits are 011 or 111 then

$$x = \sum_{t=1}^{m}(2^{i_t} - 2^{j_t}) + 2^1 - 3 (\mathrm{mod}\, n),$$

such that $k \geq j_1 > i_1 > j_2 > i_2 > ... > j_m > i_m \geq 2$. Observe that $2^1 - 3 (\mathrm{mod}\, n) = 2^k - 1$ sets ones in all bits of the binary representation of x and $2^{i_t} - 2^{j_t}$, for each $1 \leq t \leq m$, represents a block of zeroes in the binary representation of x.

First, assume that the three least significant bits are 111. A $\pm$representation of zeroes in the $k-2$ most significant bits is symmetrical to a $\pm$representation of ones in a $k-2-$bit even number. As such a $\pm$representation of length at most $\lceil k/2 \rceil$ exists, there also exists a $\pm$representation of the zeroes in the $k-2$ most significant bits of the same length. This $\pm$representation together with $2^1 - 3$ taken mod n is at most of length $\lceil k/2 \rceil + 1 = \lceil (k+2)/2 \rceil$.

Now, assume that the three least significant bits are 011. If the most significant ($k-1-$st) bit of x is 1, then our problem is similar to the previous one: a $\pm$representation of the zeroes in a $k-2-$bit odd number is symmetrical to a $\pm$representation of the ones in a $k-2-$bit even number. This $\pm$representation together with $2^1 - 3$ taken mod n is at most of length $\lceil k/2 \rceil + 1 = \lceil (k+2)/2 \rceil$.

If the most significant bit of x is 0, then the problem of representation of the $k-1$ most significant bits is symmetrical to the problem of representation of a $k-1$ bit even number with ones in the most significant bit and in the second least significant bit. In this case we can find a $\pm$representation (see Lemmas 3.2 and 3.3, parts of the proofs with application of the rule R4) where the number of gaps is at least equal to $\#(I \cup J) - 2$. The symmetrical $\pm$representation together with $2^1 - 3$ taken mod n is of length at most $\lceil k/2 \rceil + 1 = \lceil (k+2)/2 \rceil$.

4 Lower Bound

Let $0 \leq x < 2^k$ be an integer. Recall that the sequence of consecutive ones in the binary representation of x is called a block.

Lemma 4.1 *The number of blocks in $x \pm 2^i, 0 \leq i \leq k$ is by at most 1 greater than the number of blocks in x.*

Proof. Simple case analysis. □

Assume k is even. Set $x = 2^{k-1} + 2^{k-3} + 2^{k-5} + .. + 2^1$. Then x has $k/2$ blocks. Let

$$x = \sum_{t=1}^{m} (2^{i_t} - 2^{j_t}) \pmod{2^k},$$

where $1 \leq i_t, j_t \leq k, t = 1, 2, 3..., m$, be a shortest $\pm$representation of x. Notice that $I = \{i_1, i_2, ..., i_m\}$, and $J = \{j_1, j_2, ..., j_m\}$ may be multisets now.

1. If $a, a+1 \in I$ or $a, a \in I$ then $2^a + 2^{a+1} \pmod{2^k}$ and $2^a + 2^a \pmod{2^k}$ has only one block. Starting with $a, a+1$ or a, a and adding the remaining terms and noting Lemma 4.1, we can conclude that the number of blocks in x is at most $1 + 2(m-1) \geq k/2$ which gives $2m \geq (k+2)/2$. Similar analysis holds if I is replaced by J.

2. If $a \in I$ and $b \in J$, $a > b$, exist, then $2^a - 2^b$ has only one block and we continue as in the previous case.

3. If the previous two cases do not occur, then it must hold that $j_m > j_{m-1} > ... > j_1 > i_m > i_{m-1} > ... > i_1$. Then clearly

$$x = 2^k - \sum_{t=1}^{m} 2^{j_t} + \sum_{t=1}^{m} 2^{i_t} = 2^{k-1} + 2^{k-3} + 2^{k-5} + .. + 2^1.$$

   This implies that $i_t = 2t - 1$, for $t = 1, 2, ..., m$ and $2^k - \sum_{t=1}^{m} 2^{j_t} = 2^{k-1} + 2^{k-3} + ... + 2^{2m+1}$. Then $\sum_{t=1}^{m} 2^{j_t} = 2^k - (2^{k-1} + 2^{k-3} + ... + 2^{2m+1})$ contains two consecutive ones in the binary representation, a contradiction.

Let k be odd. Set $x = 2^{k-1} + 2^{k-3} + 2^{k-5} + .. + 2^2 + 1$. Then x has $(k+1)/2$ blocks. Let

$$x = (\sum_{t=1}^{m} (2^{i_t} - 2^{j_t}) + 2^{i_{m+1}} - 3) \pmod{2^k},$$

where $0 \leq i_t, j_t \leq k, t = 1, 2, 3..., m$, and $0 \leq i_{m+1} \leq k$ be a shortest representation of x. $I = \{i_1, i_1, ..., i_{m+1}\}$, and $J = \{j_1, j_1, ..., j_m\}$ may be multisets, again. Wlog assume that $i_1 \leq i_2 \leq ... \leq i_{m+1}$, and $j_1 \leq j_2 \leq ... \leq j_m$.

1. If there exists $j_s \in J$ s.t. $i_{m+1} > j_s$, then the number $2^{i_{m+1}} - 2^{j_s} + 2^{i_1} - 3$, has 3 blocks. Therefore, the total number of blocks in x is at most $3 + 2(m-1) \geq (k+1)/2$. This implies that the diameter is at least $2m + 1 \geq (k+1)/2$. Assume that the diameter equals $(k+1)/2$. Consider the number $y = 2^{k-1} + 2^{k-3} + ... + 2^2$. It has $(k-1)/2$ blocks. Let

$$y = \sum_{t=1}^{m} (2^{i_t} - 2^{j_t}) \pmod{2^k},$$

be its shortest representation. Similarly, as in the case (k even) we prove that the number of blocks in y is at most $1 + 2(m-1) \geq (k-1)/2$, which implies $2m \geq (k+1)/2$. It means that the distance between the node 0 and y and between 0 and x is $(k+1)/2$. Consider the shortest paths between x and y in the graph. The length of this path is an odd number. Then the 3 shortest paths $0-x, x-y, y-0$ produce an odd-cycle, a contradiction, as our graph is bipartite.

2. Finally assume that the previous case is not valid. If $a, a+1 \in I$ or $a, a \in I$ or $a, a+1 \in J$, or $a, a \in J$ or $a \in I, a+1 \in J$, we continue as in case k even, 1.(a) and conclude that the diameter is at least $2m+1 \geq (k+1)/2$. Suppose that the diameter is $(k+1)/2$. Consider the number y from the case k odd 1. Similarly, we prove that the distance between 0 and y is $(k+1)/2$ and force an odd cycle, a contradiction. Hence, we may assume that $j_m > j_{m-1} > ... > j_1 > i_{m+1} > i_m > i_{m-1} > ... > i_1$ and $j_1 > i_{m+1} + 1$. Then clearly

$$x = 2^k - \sum_{t=1}^{m} 2^{j_t} + \sum_{t=1}^{m} 2^{i_t} - 3 = 2^{k-1} + 2^{k-3} + 2^{k-5} + .. + 2^2 + 1.$$

This implies that $\sum_{t=1}^{m} 2^{i_t} - 3 = 2^{2m-2} + 2^{2m-4} + ... + 2^2 + 1$ and $2^k - \sum_{t=1}^{m} 2^{j_t} = 2^{k-1} + 2^{k-3} + ... + 2^{2m}$. Then $\sum_{t=1}^{m} 2^{j_t} = 2^k - (2^{k-1} + 2^{k-3} + ... + 2^{2m})$ contains two consecutive ones in the binary representation, a contradiction.

5 Conclusions

In spite of the fact that the Knödel graph is an important interconnection network and has important broadcast and gossip properties, its diameter has not yet been determined. In the presented work we have shown that the diameter of the Knödel graph $W_{k,2^k}$ is equal to $\lceil (k+2)/2 \rceil$. Note that in [11], Park and Chwa presented a new family of graphs, called *recursive circulant graphs*. In particular, they showed that the recursive circulant graph $G(2^k, 4)$, of 2^k nodes and degree k, has diameter $\lceil (3k-1)/4 \rceil$. The main reason for offering the recursive circulant graph as a new topology was its performance (number of vertices to diameter ratio) which could compete with the $k-$dimensional hypercube Q_k. Since the Knödel graph $W_{k,2^k}$ has the diameter $\lceil (k+2)/2 \rceil$, it could compete with both $G(2^k, 4)$ and Q_k.

However, the more general question of determining the diameter of the Knödel graph $W_{k,n}$ for any k is still open.

Open question is also how to find the shortest path for any pair of nodes of the Knödel graph $W_{k,2^k}$.

Finally, we note that our result (or rather proof technique) can be interpreted as a result from the area of computational arithmetic. In certain applications of modular arithmetic (for example in cryptography) it is important to quickly evaluate powers of the form y^x where y is an element of a finite group and x is an integer exponent. One can speed up the computation by expressing x as a short sum of terms of the form $\pm 2^i$. In [2] a notion of optimal representation of this type is defined and a fast (linear time) algorithm to find such representations was constructed. In fact, the authors of [2] reduce the problem of finding shortest paths in an efficiently constructible directed acyclic graph. The latter problem can then be solved fast by Dijkstra's well known algorithm. One of the consequences of our result is that the number of $\pm 2^i$ terms in such a representation of a $k-$bit x is at most $\lceil k+2/2 \rceil$.

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