High dimensional Apollonian networks

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Abstract. We propose a simple algorithm which produces high dimensional Apollonian networks with both small-world and scale-free characteristics. We derive analytical expressions for the degree distribution, the clustering coefficient and the diameter of the networks, which are determined by their dimension.

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1. Introduction

Since the ground-breaking papers by Watts and Strogatz on small-world networks [1] and Barabási and Albert on scale-free networks [2], the research interest on complex networks as an interdisciplinary subject has soared [3, 4, 5]. Complex networks describe many systems in nature and society, most of which share three apparent features: power-law degree distribution, small average path length (APL) and high clustering coefficient.

While many models [3, 4, 5] have been proposed to describe real-life networks, most of them are stochastic. However, new deterministic models with fixed degree distributions constructed by recursive methods have been recently introduced [6, 7, 8, 9, 12, 10, 11]. Deterministic models have the strong advantage that it is often possible to compute analytically their properties [13, 14], which may be compared with experimental data from real and simulated networks. Deterministic networks can be created by various techniques: modification of some regular graphs [15], addition and product of graphs [16], edge iterations [18] and other mathematical methods as in [17]. Concerning the problem of Apollonian packing, two groups independently introduced the Apollonian networks [19, 20] which have interesting properties like being scale-free, Euclidean, matching, space-filling and can be applied to porous media, polydisperse packings, road networks or electrical supply systems [19] and may also help to explain the properties of energy landscapes and the associated scale-free network of connected minima [20].

In this paper we present a simple iterative algorithm to generate high-dimensional Apollonian networks based on a similar idea as that of the recursive graphs proposed in [12]. The introduced algorithm can concretize the problems of abstract high-dimensional Apollonian packings. Using the algorithm we determine relevant characteristics of high-dimensional Apollonian networks: the degree distribution, clustering coefficient and diameter, all of which depend on the dimension of Apollonian packings.

It should be pointed out that the concept of high-dimensional Apollonian networks was already introduced in [19] and [20]. In these works, however, the emphasis is placed on two-dimensional Apollonian networks and their aim is to address the behavior of dynamical processes [19] or provide a model to help understand the energy landscape networks [20, 21, 22]. Here, we focus on the producing algorithm, based on which we provide a detailed calculation of the topology characterization of high-dimensional Apollonian networks and we show that it depends on the dimension.

2. The construction of high dimensional Apollonian networks

From the problem of Apollonian packing, a two-dimensional example of which is shown in Fig.1(a), Andrade et al. introduced Apollonian networks [19] which were independently proposed by Doye and Massen in [20]. Apollonian packing dates back to Apollonius of Perga who lived around 200 BC. The classic two-dimensional Apollonian

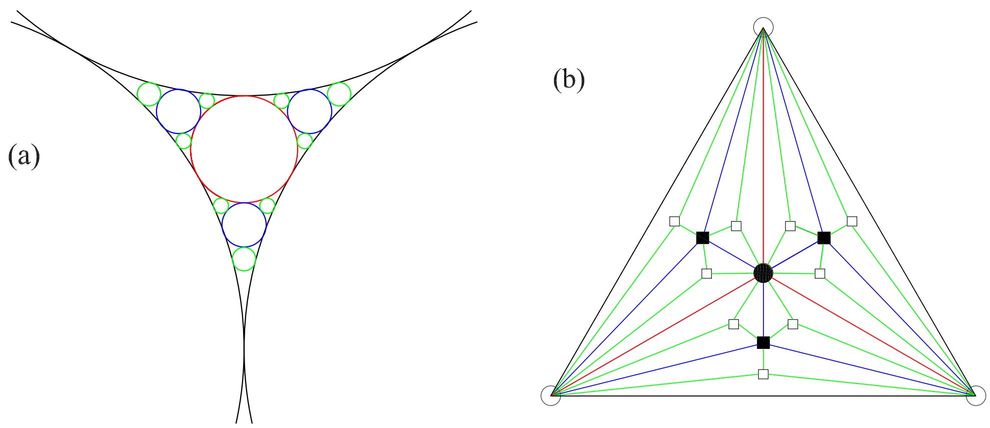

Figure 1. (a) A two-dimensional Apollonian packing of disks. (b) Construction of two-dimensional Apollonian networks, showing the first four iterations steps.

packing is constructed by starting with three mutually touching circles, whose interstice is a curvilinear triangle to be filled. Then a circle is inscribed, touching all the sides of this curvilinear triangle. We call this the first iteration $t = 1$ and the initial configuration is denoted by $t = 0$. For subsequent iterations we indefinitely repeat the process for all the newly generated curvilinear triangles. In the limit of infinite iterations, the well-known two-dimensional Apollonian packing is obtained.

From the two-dimensional Apollonian packing, one can straightforwardly define a two-dimensional Apollonian network [19, 20], where vertices are associated to the circles and two vertices are connected if the corresponding circles are tangent. Fig.1(b) shows the network based on the two-dimensional Apollonian packing. The two-dimensional Apollonian network can be generalized to high-dimensions (d-dimensional, $d \geq 2$) [20] associated with other self-similar packings [23]. A comprehensive account is given next.

A d-dimensional ($d > 2$) Apollonian packing can be constructed iteratively in a similar way as shown in Fig. 1(a). Initially, we have $d + 1$ mutually touching d-dimensional hyperspheres with a curvilinear open d-dimensional polyhedron (d-polyhedron) as their interstice. In the first iteration one d-hypersphere is added to fill the interstice of the initial configuration, such that it should touch each of the $d+1$ d-hyperspheres. The process is repeated for all the newly created curvilinear open d-dimensional polyhedrons in the successive iterations. In the limit of infinite iterations, the result is a d-dimensional Apollonian packing. If each d-hypersphere corresponds to a vertex and vertices are connected by an edge if the corresponding d-hyperspheres are in contact, then one gets a d-dimensional Apollonian network.

3. The iterative algorithm of high dimensional Apollonian networks

In the iterative process for the construction of high-dimensional Apollonian networks at each iteration, for each new hypersphere added, $d+1$ new interstices are created in the associated Apollonian packing which will be filled in the next iteration. when building networks, we can say in equivalent words that for each new vertex added, $d+1$ new d-polyhedrons are generated in the network, into which vertices will be inserted in the next iteration. According to this process, we can introduce a general iterative algorithm to create high-dimensional Apollonian networks which is similar to the process that allow the construction of the recursive graphs introduced in [12].

Before introducing the algorithm we give the following definitions. A *complete graph* K_d (also referred in the literature as *d-clique*; see [24]) is a graph with d vertices, where there is no loop or multiple edge and every vertex is joined to every other by an edge. Generally speaking, two graphs are said to be *isomorphic* if the vertices and edges of one graph match up with vertices and edges of other, and the edge matching be consistent with the vertex matching.

We denote the d-dimensional Apollonian network after t iterations by $A(d,t)$, $d \geq 2, t \geq 0$. Then the d-dimensional Apollonian network at step t is constructed as follows: For $t = 0$, $A(d,0)$ is the complete graph K_{d+1} (or $(d+1)$-clique), and $A(d,0)$ has $d+1$ vertices and $\frac{(d+1)d}{2}$ edges. For $t \geq 1$, $A(d,t)$ is obtained from $A(d,t-1)$ by adding for each of its existing subgraphs isomorphic to a $(d+1)$-clique and created at step $t-1$ a new vertex and joining it to all the vertices of this subgraph (see Fig. 1(b) for the case $d = 2$). Then, at $t = 1$, we add one new vertex and $d+1$ new edges to the graph, creating $d+1$ new K_{d+1} cliques and resulting in the complete graph with $d+2$ vertices, denoted K_{d+2}. At $t = 2$ we add $d+1$ new vertices, each of them connected to all the vertices of one of the $d+1$ cliques K_{d+1} created at $t = 1$ introducing $(d+1)^2$ new edges, and so on.

Note that the addition of each new vertex leads to $d+1$ new $(d+1)$-cliques and $d+1$ new edges. So the number of new vertices and edges at step t_i is $L_v(t_i) = (d+1)^{t_i-1}$ and $L_e(t_i) = (d+1)^{t_i}$, respectively. Therefore, similarly to many real-life networks such as the World Wide Web, the d-dimensional Apollonian network is a growing network, whose number of vertices increases exponentially with time.

Thus we can easily see that at step t, the Apollonian network $A(d,t)$ has

$$N_t = (d+1) + \sum_{t_i=1}^{t} L_v(t_i) = \frac{(d+1)^t - 1}{d} + d + 1 \tag{1}$$

vertices and

$$|E|_t = \frac{d(d+1)}{2} + \sum_{t_i=1}^{t} L_e(t_i) = \frac{d(d+1)}{2} + \frac{(d+1)^{t+1} - d - 1}{d} \tag{2}$$

edges

The average degree $\overline{k}_t$ is then

$$\overline{k}_t = \frac{2|E|_t}{N_t} = \frac{2(d+1)^{t+1} + d^3 + d^2 - 2d - 2}{(d+1)^t + d^2 + d - 1} \tag{3}$$

For large t it is approximately $2(d+1)$. We can see when t is large enough the resulting networks are sparse graphs as many real-world networks whose vertices have many fewer connections than is possible.

4. Relevant characteristics of high dimensional Apollonian networks

Below we will find that the dimension d is a tunable parameter controlling all the relevant characteristics of the d-dimensional Apollonian network.

4.1. Degree distribution

When a new vertex i is added to the graph at step t_i ($t_i \geq 1$), it has degree $d+1$ and forms $d+1$ new $(d+1)$-cliques. From the iterative algorithm, we can see that each new neighbor of i generated d new $(d+1)$-cliques with i as one vertex of them. In the next iteration, these $(d+1)$-cliques will introduce new vertices that are connected to the vertex i. Let $k_i(t)$ be the degree of i at step t ($t > t_i + 1$). Then, as in [20],

$$\Delta k_i(t) = k_i(t) - k_i(t-1) = d\Delta k_i(t-1) \tag{4}$$

combining the initial condition $k_i(t_i) = d+1$ and $\Delta k_i(t_i+1) = d+1$, we obtain

$$\Delta k_i(t) = (d+1)d^{t-t_i-1} \tag{5}$$

and the degree of vertex i becomes

$$k_i(t) = k_i(t_i) + \sum_{t_m=t_i+1}^{t} \Delta k_i(t_m) = (d+1)\left(\frac{d^{t-t_i}-1}{d-1}+1\right) \tag{6}$$

The distribution of all vertices and their degrees at step t is given in Table 1. It should be mentioned that here we don't give the detailed degree evolution process of the $d+1$ initial vertices created at step 0 and just list the evolution result, which is different form others vertices. But when the network become very large, these few initial vertices have almost nothing effect on the network topology characteristics. From Table 1 we can see that the degree spectrum of the graph is discrete and some values of the degree are absent. To relate the exponent of this discrete degree distribution to the standard γ exponent as defined for continuous degree distribution, we use a cumulative distribution $P_{cum}(k) \equiv \sum_{k' \geq k} N(k',t)/N_t \sim k^{1-\gamma}$. Here k and k' are points of the discrete degree spectrum. The analytic computation details are given as follows.

For a degree k

$$k = (d+1)\left(\frac{d^{t-l}-1}{d-1}+1\right)$$

there are $(d+1)^{l-1}$ vertices with this exact degree, all of which were born at step l.

Table 1. Distribution of vertices and their degrees for $A(d,t)$ at step t.

Number of vertices	Degree
$d+1$	$\sum_{j=0}^{t-1} d^j + d$
1	$(d+1)(\sum_{j=0}^{t-2} d^j + 1)$
$d+1$	$(d+1)(\sum_{j=0}^{t-3} d^j + 1)$
$(d+1)^2$	$(d+1)(\sum_{j=0}^{t-4} d^j + 1)$
...	...
$(d+1)^{t-3}$	$(d+1)(d+1+1)$
$(d+1)^{t-2}$	$(d+1)(d+1)$
$(d+1)^{t-1}$	$d+1$

All vertices introduced at time l or earlier have this and a higher degree. So we have

$$\sum_{k' \geq k} N(k',t) = (d+1) + \sum_{s=1}^{l} L_v(s) = \frac{(d+1)^l - 1}{d} + d + 1$$

As the total number of vertices at step t is given in Eq. (1) we have

$$\left[(d+1)(\frac{d^{t-l}-1}{d-1}+1)\right]^{1-\gamma} = \frac{\frac{(d+1)^l-1}{d}+d+1}{\frac{(d+1)^t-1}{d}+d+1}$$

$$= \frac{(d+1)^l + d(d+1) - 1}{(d+1)^t + d(d+1) - 1} \qquad (7)$$

Therefore, for large t we obtain

$$(d^{t-l})^{1-\gamma} = (d+1)^{l-t}$$

and

$$\gamma \approx 1 + \frac{\ln(d+1)}{\ln d} \qquad (8)$$

so that $2 < \gamma < 2.58496$.

We notice that this value has been obtained previously by Doye and Massen in [20]. Also, notice that when t gets large, the maximal degree of a vertex roughly equals to $d^{t-1} \sim N_t^{\ln d/\ln(d+1)} = N_t^{1/(\gamma-1)}$.

4.2. Clustering distribution

The clustering coefficient [1] of a given vertex is the ratio of the total number of edges that actually exist between all its k nearest neighbors and the potential number of edges $k(k-1)/2$ between them. The clustering coefficient of the whole network is obtained averaging over all its vertices. We can derive analytical expressions for the clustering $C(k)$ for any vertex with degree k.

When a vertex is created it is connected to all the vertices of a $(d+1)$-clique whose vertices are completely interconnected. It follows that a vertex with degree $k = d+1$ has a clustering coefficient of one because all the $(d+1)d/2$ possible links between its neighbors actually exist. After that, if its degree increases by one, then its new neighbor must link to its d existing neighbors. Thus for a vertex v of degree k, the exact expression for its clustering coefficient is

$$C(k) = \frac{\frac{d(d+1)}{2} + d(k-d-1)}{\frac{k(k-1)}{2}} = \frac{2d(k-\frac{d+1}{2})}{k(k-1)} \tag{9}$$

depending on degree k and dimension d. Using this result, we can compute now the clustering of the graph at step t, it is $\overline{C}_t = S_t/N_t$, where N_t is number of vertices at step t which is provided by Eq. (1) and S_t represents the sum of clustering coefficient for all vertices given by

$$S_t = (d+1)\frac{2d(D_0 - \frac{d+1}{2})}{D_0(D_0-1)} + \sum_{q=1}^{t} \frac{2d(D_q - \frac{d+1}{2})L_v(q)}{D_q(D_q-1)} \tag{10}$$

where $D_0 = \frac{d^t-1}{d-1} + d$ and $D_q = (d+1)\left(\frac{d^{t-q}-1}{d-1}+1\right)$ given by Eq. (6) are the degrees of the vertices created at steps 0 and q, respectively. One can easily prove that for $t \geq 7$ and for any $d \geq 2$ the following relation hold true.

$$\overline{C}_t > \frac{3d-2}{3d-1} \tag{11}$$

Therefore the clustering coefficient of high dimension Apollonian networks is very large. Similarly to the power exponent γ of the degree distribution, the clustering is also tunable simply by changing the value of control parameter d. From Eq. (11), one can see that the clustering coefficient increases with d and approaches a limit of 1 when d gets large. In the special cases $d = 2$ and $d = 3$, $\overline{C}$ equals to constant asymptotic values 0.8284 (see also [19]) and 0.8852, respectively.

4.3. Diameter

The diameter of a network characterizes the maximum communication delay in the network and is defined as the longest shortest path between all pairs of vertices. In what follows, the notations $\lceil x \rceil$ and $\lfloor x \rfloor$ express the integers obtained by rounding x to the nearest integers towards infinity and minus infinity, respectively. Now we compute the diameter of $A(d,t)$, denoted $diam(A(d,t))$ for $d \geq 2$:

Step 0 and 1. The diameter is 1.

Steps 2 to $\lceil \frac{d}{2} \rceil + 1$. In this case, the diameter is 2, since any new vertex is by construction connected to a $(d+1)$-clique, and since any $(d+2)$-clique during those steps contains at least the vertex created at step 1, which is from the initial clique K_{d+2} or $A(d,1)$ obtained after step 1, thus the diameter is 2.

Steps $\lceil \frac{d}{2} \rceil + 2$ *to* $d+2$. In any of those steps, some newly added vertices might not share a neighbor in the original clique K_{d+2} obtained after step 1; however, any newly added vertex is connected to at least one vertex of the initial clique K_{d+2}. Thus, the diameter is equal to 3.

Further steps. Clearly, at each step $t \geq d+3$, the diameter always lies between a pair of vertices that have just been created at this step. We will call the newly created vertices "outer" vertices. At any step $t \geq d+3$, we note that an outer vertex cannot be connected with two or more vertices that were created during the same step $0 < t' \leq t-1$. Moreover, by construction no two vertices that were created during a given step are neighbors, thus they cannot be part of the same $(d+2)$-clique. Thus, for any step $t \geq d+3$, some outer vertices are connected with vertices that appeared at pairwise different steps. Thus, there exists an outer vertex v_t created at step t, which is connected to vertices v_i, $1 \leq i \leq t-1$, which all are pairwise distinct. We conclude that v_t is necessarily connected to a vertex that was created at a step $t_0 \leq t-d-1$. If we repeat this argument, then we obtain an upper bound on the distance from v_t to the initial clique K_{d+2}. Let $t = \alpha(d+1)+p$, where $2 \leq p \leq d+2$. Then, we see that v_t is at distance at most $\alpha+1$ from a vertex in K_{d+2}. Hence any two vertices v_t and w_t in $A(d,t)$ lie at distance at most $2(\alpha+1)+1$; however, depending on p, this distance can be reduced by 1, since when $p \leq \lceil \frac{d}{2} \rceil + 1$, we know that two vertices created at step p share at least a neighbor in K_{d+2}. Thus, when $2 \leq p \leq \lceil \frac{d}{2} \rceil + 1$, $diam(A(d,t)) \leq 2(\alpha+1)$, while when $\lceil \frac{d}{2} \rceil + 2 \leq p \leq d+2$, $diam(A(d,t)) \leq 2(\alpha+1)+1$. One can see that these distance bounds can be reached by pairs of outer vertices created at step t. More precisely, those two vertices v_t and w_t share the property that they are connected to d vertices that appeared respectively at steps $t-1, t-2, \ldots t-d-1$.

Based on the above arguments, one can easily see that for $t > d+2$, the diameter increases by 2 every $d+1$ steps. More precisely, we have the following result, for any $d \geq 2$ and $t \geq 2$ (when $t = 1$, the diameter is clearly equal to 1):

$$diam(A(d,t)) = 2(\lfloor \frac{t-2}{d+1} \rfloor + 1) + f(d,t)$$

where $f(d,t) = 0$ if $t - \lfloor \frac{t-2}{d+1} \rfloor (d+1) \leq \lceil \frac{d}{2} \rceil + 1$, and 1 otherwise.

In the limit of large t, $diam(A(d,t)) \sim \frac{2t}{d+1}$, while $N_t \sim (d+1)^{t-1}$, thus the diameter is small and scales logarithmically with the network size.

5. Conclusion and discussion

In conclusion, we have proposed a general iterative algorithm to produce high dimensional Apollonian networks associated with high dimensional packings. The networks present the typical characteristics of real-life networks in nature and society as they are small-world and have a power-law degree distribution. We compute analytical expressions for the degree distribution, the clustering coefficient, and the diameter of the networks, all of which are determined by the dimension of the associated Apollonian packings. The high dimensional Apollonian networks introduced here, and the consideration of the method presented in Ref. [25], allow the construction of high dimensional random Apollonian networks [26]. In addition, it should be worth studying in detail physical models such as Ising models [27] and processes such as percolation, spreading, searching and diffusion that take place on the higher-dimensional Apollonian networks to know also their relation with the dimension.

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